...

Application of Log-linearization Methods: Optimal Policy

Log-linear Methods

- Equilibrium conditions:

$$v\left(k_t, k_{t+1}, k_{t+2}\right) = 0,$$

  $t = 0, 1, 2, ...$

- Solution:
  - compute steady state, k^* such that $v\left(k^*, k^*, k^*\right) = 0$.
  - expansion about steady state: $V_0 \tilde{k}_t + V_1 \tilde{k}_{t+1} + V_2 \tilde{k}_{t+2} = 0$.
  - solve linearized system.

Log-linear Methods ...

– what is optimal monetary policy?

– drop monetary policy rule

– now we're short one equation!

– system underdetermined....'many solutions'

– pick the best one.

- Potential problem: time inconsistency of optimal monetary policy:
  - period t announcement about period $t+1$ policy action, X, influenced in part by the impact of X on period t decisions by the public.
  - when $t+1$ occurs and it is time to actually implement X, period t decisions by public are past history.
    - temptation in $t+1$ to modify X since X no longer influences period t decisions of public.
  - temptation to modify X in $t+1$ must be avoided, if there is to be any hope to have optimal policy. Bad outcomes could occur otherwise.
    - discipline on the part of policy makers is required, if they are to avoid temptation to deviate.
- Technical implication of potential time inconsistency.
  - v equilibrium conditions seemingly not time invarient: apparently our log-linearization methods do not apply!
  - follow Kydland-Prescott 'trick' and put problem in Lagrangian form.
  - problem of avoiding temptation to deviate boils down to the admonition, 'remember your multipliers!'

Example #1: Optimal Monetary Policy - Toy Example

- Setup
  - Model
    - One equation characterizing private sector behavior:

$$\pi_t - \beta\pi_{t+1} - \gamma y_t = 0, \ t = 0, 1, 2, ... \tag{1}$$

    - Another equation characterizes policy.
  - Want to do *optimal* policy, so throw away policy equation.
  - System is now under-determined: one equation in two variables, π_t and y_t.

Example #1: Optimal Monetary Policy - Toy Example ...

– Optimization delivers the other equations.

∗ optimize objective:

$$\sum_{t=0}^{\infty} \beta^t u(\pi_t, y_t)$$

subject to (1).

∗ If objective corresponds to social welfare function, this is called *Ramsey* optimal problem

∗ Objective may be preferences of policy maker.

Example #1: Optimal Monetary Policy - Toy Example ...

- Lagrangian representation of problem:

$$\max_{\{\pi_t, y_t; t=0,1,...\}} \sum_{t=0}^{\infty} \beta^t \{u(\pi_t, y_t) + \lambda_t [\pi_t - \beta\pi_{t+1} - \gamma y_t]\}$$

$$= \max_{\{\pi_t, y_t; t=0,1,...\}} \{u(\pi_0, y_0) + \lambda_0 [\pi_0 - \beta\pi_1 - \gamma y_0]$$

$$+\beta u(\pi_1, y_1) + \beta\lambda_1 [\pi_1 - \beta\pi_2 - \gamma y_1] + ...\}$$

- First order necessary conditions for optimization:

$$\begin{aligned} u_\pi(\pi_0, y_0) + \lambda_0 &= 0 \text{ (*)} \\ u_\pi(\pi_1, y_1) + \lambda_1 - \lambda_0 &= 0 \\ &\dots \\ u_y(\pi_0, y_0) - \gamma\lambda_0 &= 0 \\ u_y(\pi_1, y_1) - \gamma\lambda_1 &= 0 \\ &\dots \\ \pi_0 - \beta\pi_1 - \gamma y_0 &= 0 \\ \pi_1 - \beta\pi_2 - \gamma y_1 &= 0 \\ &\dots \end{aligned}$$

Example #1: Optimal Monetary Policy - Toy Example ...

- These equations 'look' different than the ones we've seen before
  - They are not stationary, (*) is different from the others.
    - reflects that at time 0 there is a constraint 'missing'
    - no need to respect what people were expecting you to do as of time -1
    - do need to respect what they expect you to do in the future, because that affects current behavior.
    - that's the source of the 'time inconsistency of optimal plans'.
- Can trick the problem into being stationary (see, e.g., Kydland and Prescott (JEDC, 1990s) and Levin, Onatski, Williams, and Williams, Macro Annual, 2005). Then, apply standard log-linearization solution method.

Example #1: Optimal Monetary Policy - Toy Example ...

- Consider:

$$v(\pi_t, \pi_{t+1}, y_t, \lambda_t, \lambda_{t-1}) = \begin{bmatrix} u_\pi(\pi_t, y_t) + \lambda_t - \lambda_{t-1} \\ u_y(\pi_t, y_t) - \gamma\lambda_t \\ \pi_t - \beta\pi_{t+1} - \gamma y_t \end{bmatrix}, \text{ for all } t.$$

  – time t 'endogenous variables': λ_t, π_t, y_t

  – time t 'state variable': λ_{t-1}.

  – 'solution':

$$\lambda_t = \lambda(\lambda_{t-1}), \; \pi_t = \pi(\lambda_{t-1}), \; y_t = y(\lambda_{t-1}),$$

  such that

$$v(\pi(\lambda_{t-1}), \pi(\lambda(\lambda_{t-1})), y(\lambda_{t-1}), \lambda(\lambda_{t-1}), \lambda_{t-1}) = 0, \text{ for all possible } \lambda_{t-1}.$$

Example #1: Optimal Monetary Policy - Toy Example ...

- In general, solving this problem exactly is intractable.
- But, can log-linearize!

  – **Step 1**: find π^*, y^*, λ^* such that following three equations are satisfied:
  $$v\left(\pi^*, \pi^*, y^*, \lambda^*, \lambda^*\right) = \underbrace{0}_{3\times 1}.$$

  – **Step 2**: log-linearly expand v about steady state
  $$v\left(\pi_t, \pi_{t+1}, y_t, \lambda_t, \lambda_{t-1}\right) \simeq v_1\pi^*\hat{\pi}_t + v_2\pi^*\hat{\pi}_{t+1} + v_3 y^*\hat{y}_t + v_4\Delta\hat{\lambda}_t + v_5\Delta\hat{\lambda}_{t-1},$$
  where
  $\Delta\hat{\lambda}_t \equiv \lambda_t - \lambda^*$ (play it safe, don't divide by something that could be zero!)

  – **Step 3**: Posit
  $$\Delta\hat{\lambda}_t = A_\lambda \Delta\hat{\lambda}_{t-1},\ \hat{\pi}_t = A_\pi \Delta\hat{\lambda}_{t-1},\ \hat{y}_t = A_y \Delta\hat{\lambda}_{t-1},$$
  and find A_λ, A_π, A_y that solve
  $$\left[v_1\pi^* A_\pi + v_2\pi^* A_\pi A_\lambda + v_3 y^* A_y + v_4 A_\lambda + v_5\right]\Delta\hat{\lambda}_{t-1} = \underbrace{0}_{3\times 1}$$
  for all $\Delta\hat{\lambda}_{t-1}$.

Example #1: Optimal Monetary Policy - Toy Example ...

- What does the stationary solution have to do with the original non-stationary problem?
  - Do we have a solution to the period 0 problem, (*)?

$$u_{\pi}\left(\pi_0, y_0\right) + \lambda_0 = 0.$$

  - Yes! Just pretend that this equation really has the following form:

$$u_{\pi}\left(\pi_0, y_0\right) + \lambda_0 - \lambda_{-1} = 0.$$

Expression (*) does have this form, if we set $\lambda_{-1} = 0$. Then,

$$\pi_0 = \pi\left(0\right),\ y_0 = y\left(0\right),\ \lambda_0 = \lambda\left(0\right).$$

- The situation is exactly what it is in the neoclassical model when we want to know what happens when initial capital is away from steady state.

  – Plug k_0 into the stationary rule

$$k_1 = g(k_0).$$

- Possible computational pitfall: if $\lambda_{-1} = 0$ is far from λ^*, then linearized solution might be highly inaccurate

Example #1: Optimal Monetary Policy - Toy Example ...

- Optimal policy in real time.

- Suppose today is date zero.

  - Solve for $\lambda(\cdot), y(\cdot), \pi(\cdot)$

  - set $\lambda_{-1} = 0$

  - Compute and present in charts:

$$
\begin{aligned}
\lambda_0 &= \lambda(\lambda_{-1}),\ y_0 = y(\lambda_{-1}),\ \pi_0 = \pi(\lambda_{-1}) \\
\lambda_1 &= \lambda(\lambda_0),\ y_1 = y(\lambda_0),\ \pi_1 = \pi(\lambda_0) \\
&\ldots \\
\lambda_t &= \lambda(\lambda_{t-1}),\ y_t = y(\lambda_{t-1}),\ \pi_t = \pi(\lambda_0) \\
&\ldots.
\end{aligned}
$$

Example #1: Optimal Monetary Policy - Toy Example ...

- The optimal policy program may break down if policy makers succumb to the temptation to restart the Ramsey problem at a later date.

  - there is a temptation in period 1 when π_1 is determined, to ignore a constraint that went into determining the announcement made about π_1 in period 0:

$$\pi_0 - \beta\pi_1 - \gamma y_0 \text{ (*)}$$

  - If (*) is ignored at date 1, then π_1 computed in date 1 solves a different problem than π_1 computed at date 0 and there will be time inconsistency.

Example #1: Optimal Monetary Policy - Toy Example ...

- Honoring past announcements is equivalent to 'always respect the past multipliers'.
  - 'Remembering λ_0' in period 1 ensures that constraint

$$\pi_0 - \beta\pi_1 - \gamma y_0 \text{ (*)}$$

    is incorporated in period 1. In this case, π_1 solves the same problem in period 1 that it did in period 0.

- Practical implication of the admonition, 'always respect your multipliers':
  - Charts released after later meetings will be consistent with the continuation of charts released after later meetings.

Example #1: Optimal Monetary Policy - Toy Example ...

– Example:

date 0 meeting : $y_0 = y(0)\,,\ y_1 = y(\lambda(\lambda_{-1}))\,, y_2 = y(\lambda(\lambda(\lambda_{-1})))\,, \ldots$

date 1 meeting :
YES - $y_1 = y(\lambda(\lambda_{-1}))\,,\ y_2 = y(\lambda(\lambda(\lambda_{-1})))\,, \ldots$
NO - $y_1 = y(0)\,,\ y_2 = y(\lambda_1(0))\,, \ldots$

– If Central Bank selects the bad **(‘NO’)** option people will see the temporal inconsistency of policy, and CB will lose credibility.

– Any differences in charts from one meeting to the next must be fully explicable in terms of new information.

Example #2: Optimal Monetary Policy - More General Discussion

- The equilibrium conditions of a model

$$E_t \underbrace{f\left(z_{t-1}, z_t, z_{t+1}, s_t, s_{t+1}\right)}_{(N-1)\times 1} = 0, \text{ for all } \underbrace{z_{t-1}}_{N\times 1} \text{ (endogenous), } s_t \text{ (exogenous)}$$

$$s_t = P s_{t-1} + \varepsilon_t.$$

- Preferences:

$$E_t \sum_{t=0}^{\infty} \beta^t U\left(z_t, s_t\right).$$

  – Could include discounted utility in f :

$$v\left(z_{t-1}, z_t, s_t\right) = U\left(z_t, s_t\right) + \beta E_t v\left(z_t, z_{t+1}, s_{t+1}\right)$$

Example #2: Optimal Monetary Policy - More General Discussion ...

- Optimum problem:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left\{ U\left(z_t, s_t\right) + \underbrace{\lambda_t'}_{1\times(N-1)} E_t \underbrace{f\left(z_{t-1}, z_t, z_{t+1}, s_t, s_{t+1}\right)}_{(N-1)\times 1} \right\}.$$

- N first order conditions:

$$\begin{aligned} &\underbrace{U_1\left(z_t, s_t\right)}_{1\times N} + \underbrace{\lambda_t'}_{1\times(N-1)} E_t \underbrace{f_2\left(z_{t-1}, z_t, z_{t+1}, s_t, s_{t+1}\right)}_{(N-1)\times N} \\ &\quad + \beta^{-1} \underbrace{\lambda_{t-1}'}_{1\times(N-1)} \underbrace{f_3\left(z_{t-2}, z_{t-1}, z_t, s_{t-1}, s_t\right)}_{(N-1)\times N} \\ &\quad + \beta \underbrace{\lambda_{t+1}'}_{1\times(N-1)} E_t \underbrace{f_1\left(z_t, z_{t+1}, z_{t+2}, s_{t+1}, s_{t+2}\right)}_{(N-1)\times N} = \underbrace{0}_{1\times N} \end{aligned}$$

  – Endogenous variables: $z_t\ (N)$, $\lambda_t\ (N-1)$

  – Equations: Ramsey optimality conditions (N), equilibrium condition $(N-1)$

Example #2: Optimal Monetary Policy - More General Discussion ...

- First order conditions of optimum problem have exactly the same form as the type of problem we solved using linearization methods.

  – must differentiate f (includes private first order conditions that have already involved differentiation!)

  – good news:
    Dynare code for solving the system

Optimal Monetary Policy - CGG

$$
\begin{aligned}
&\max_{\nu_t, p_t^*, N_t, R_t, \bar{\pi}_t, F_t, K_t} E_0 \sum_{t=0}^{\infty} \beta^t \{ \left(\log N_t + \log p_t^* - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right) \\
&+\lambda_{1t} \left[\frac{1}{p_t^* N_t} - E_t \frac{A_t \beta}{p_{t+1}^* A_{t+1} N_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \right] \\
&+\lambda_{2t} \left[\frac{1}{p_t^*} - \left((1-\theta) \left(\frac{1-\theta(\bar{\pi}_t)^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right) \right] \\
&+\lambda_{3t} \left[1 + E_t \bar{\pi}_{t+1}^{\varepsilon-1} \beta \theta F_{t+1} - F_t \right] \\
&+\lambda_{4t} \left[(1-\nu_t) \frac{\varepsilon}{\varepsilon-1} \exp(\tau_t) N_t^{1+\varphi} p_t^* (1-\psi+\psi R_t) + E_t \bar{\pi}_{t+1}^{\varepsilon} \beta \theta K_{t+1} - K_t \right] \\
&+\lambda_{5t} \left[F_t \left[\frac{1-\theta \bar{\pi}_t^{\varepsilon-1}}{1-\theta} \right]^{\frac{1}{1-\varepsilon}} - K_t \right] \}
\end{aligned}
$$

- ‘two degree of freedom’ 7 variables, 5 equilibrium conditions

- Law of motion of technology:

$$A_t = \rho A_{t-1} + u_t.$$

- We only consider the case,

$$(1-\nu)\frac{\varepsilon}{\varepsilon-1} = 1.$$

- First consider the case, $\psi = 0$

  – Conjecture: restrictions 1, 3, 4, 5 nonbinding (i.e., $\lambda_{1t} = \lambda_{3t} = \lambda_{4t} = \lambda_{5t} = 0$)

    * Step 1: Optimize w.r.t. p_t^*, $\bar{\pi}_t$, N_t ignoring restrictions 1, 3, 4, 5.

    * Step 2: Solve for ν_t, R_t, F_t, K_t, to satisfy restrictions 1, 3, 4, 5.

  – If this can be done, then the conjecture is verified.

- Simplified problem under conjecture:

$$\max_{\bar{\pi}_t, p_t^*, N_t} E_0 \sum_{t=0}^{\infty} \beta^t \{ \left(\log N_t + \log p_t^* - \exp\left(\tau_t\right) \frac{N_t^{1+\varphi}}{1+\varphi} \right)$$

$$+\lambda_{2t} \left[\frac{1}{p_t^*} - \left((1-\theta) \left(\frac{1-\theta\left(\bar{\pi}_t\right)^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right) \right] \}$$

- Bottom line. Optimality under state-contingent ν_t implies:

$$
\begin{aligned}
p_t^* &= \left[(1-\theta)+\theta\left(p_{t-1}^*\right)^{(\varepsilon-1)}\right]^{\frac{1}{(\varepsilon-1)}} \\
\bar{\pi}_t &= \frac{p_{t-1}^*}{p_t^*} \\
N_t &= \exp\left(-\frac{\tau_t}{1+\varphi}\right) \\
1-\nu &= \frac{\varepsilon-1}{\varepsilon} \\
C_t &= p_t^* A_t N_t .
\end{aligned}
$$

- Ramsey-optimal policy is time consistent (no forward-looking constraints on core problem).
- If $\psi > 0$ and ν_t not state-contingent must work out Ramsey solution numerically.

- Example - no working capital channel ($\psi = 0$):

$$\theta = 0.75,\ \varepsilon = 2,\ \beta = 0.99,\ \rho = 0.5,\ \varphi = 1.$$

- In this case:

$$\begin{aligned}
N_t &= 1 + 0.45(\lambda_{1t-1} - \lambda_1) + .06(\lambda_{3,t-1} - \lambda_3) + 0.63(\lambda_{4,t-1} - \lambda_4) \\
r_t &= 0.01 - 0.50(\lambda_{1t-1} - \lambda_1) + 0.10(\lambda_{3,t-1} - \lambda_3) - 0.02(\lambda_{4,t-1} - \lambda_4) - 0.25a_{t-1} \\
&\quad -0.51u_t \\
\pi_t &= 1 + 0.07(\lambda_{1t-1} - \lambda_1) + 0.09(\lambda_{3,t-1} - \lambda_3) + 0.31(\lambda_{4,t-1} - \lambda_4) + 0.25(p^*_{t-1} - 1) \\
\lambda_{1t} &= 0, \\
\lambda_{2,t} &= 3.88 + 0.82(\lambda_{1t-1} - \lambda_1) + 1.46(\lambda_{3,t-1} - \lambda_3) + 3.65(\lambda_{4,t-1} - \lambda_4) \\
&\quad +4.13(p^*_{t-1} - 1) \\
\lambda_{3,t} &= 0.05(\lambda_{1t-1} - \lambda_1) + 0.69(\lambda_{3,t-1} - \lambda_3) + 0.12(\lambda_{4,t-1} - \lambda_4) \\
\lambda_{4,t} &= -0.05(\lambda_{1t-1} - \lambda_1) + 0.06(\lambda_{3,t-1} - \lambda_3) + 0.63(\lambda_{4,t-1} - \lambda_4) \\
\lambda_{5,t} &= 0.05(\lambda_{1t-1} - \lambda_1) - 0.06(\lambda_{3,t-1} - \lambda_3) + 0.12(\lambda_{4,t-1} - \lambda_4)
\end{aligned}$$

$$\lambda_1 = \lambda_3 = \lambda_4 = \lambda_5 = 0, \lambda_2 = 3.88$$

- ‘Resetting multipliers’ makes no difference: *no* time inconsistency problem.

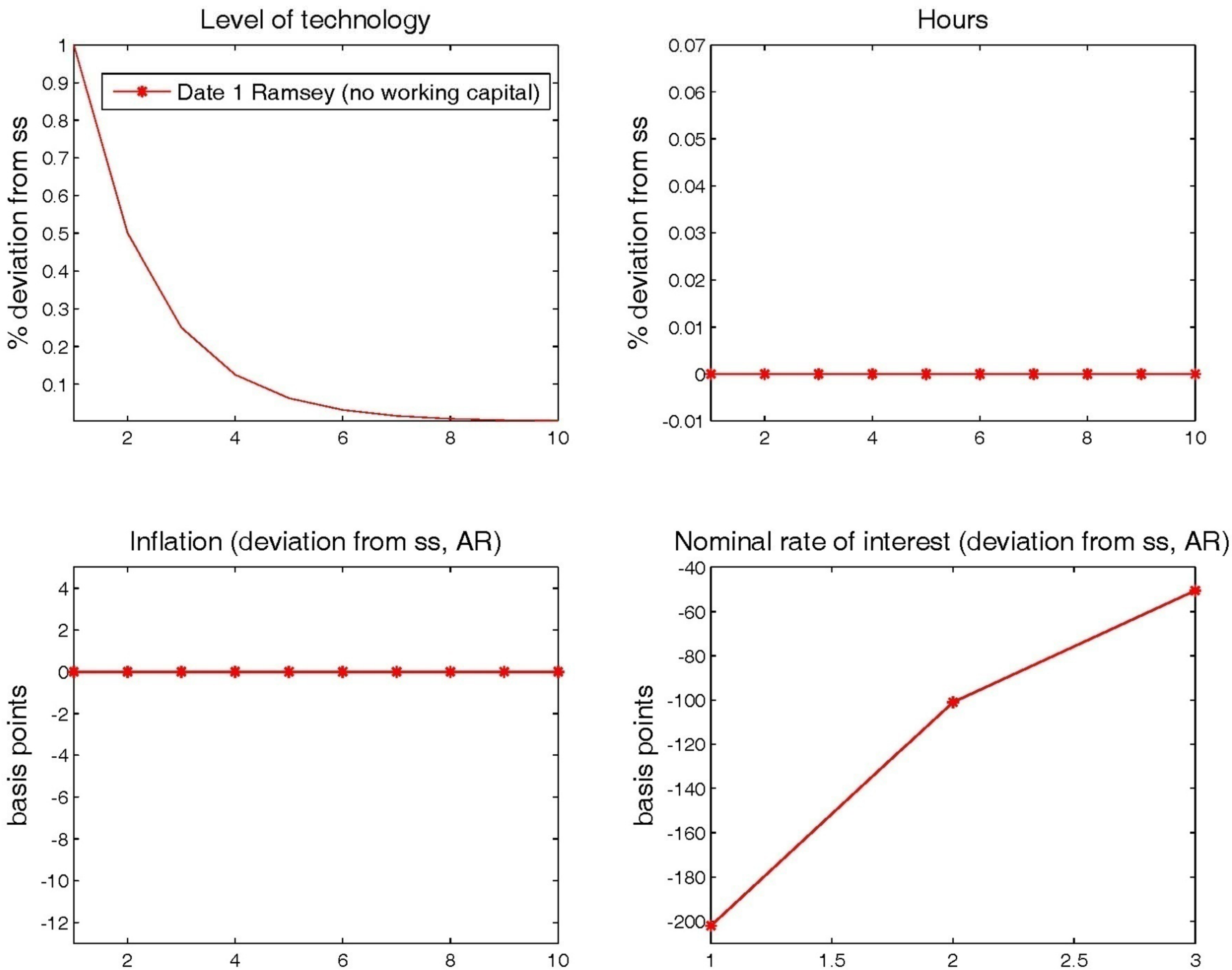
- Example with $\psi = 0.7$:

$$
\begin{aligned}
N_t &= 1 + 0.50\lambda_{1t-1} + .03\lambda_{3,t-1} + 0.40\lambda_{4,t-1} + 0.02a_{t-1} + 0.03u_t \\
r_t &= 0.01 - 0.51\lambda_{1t-1} + 0.12\lambda_{3,t-1} + 0.30\lambda_{4,t-1} - 0.24a_{t-1} - 0.49u_t \\
\pi_t &= 1 + 0.05\lambda_{1t-1} + 0.10\lambda_{3,t-1} + 0.31\lambda_{4,t-1} - 0.01a_{t-1} + 0.25(p^*_{t-1} - 1) - 0.02u_t \\
p^*_t &= 1 + .75(p^*_{t-1} - 1) \\
\lambda_{1t} &= -0.01\lambda_{1t-1} + 0.04\lambda_{3,t-1} + 0.44\lambda_{4,t-1} + 0.02A_{t-1} + 0.03u_t \\
\lambda_{2,t} &= 3.88 + 0.95\lambda_{1t-1} + 1.42\lambda_{3,t-1} + 3.63\lambda_{4,t-1} + 0.09A_{t-1} + 0.18u_t + 4.13(p^*_{t-1} - 1) \\
\lambda_{3,t} &= 0.01\lambda_{1t-1} + 0.70\lambda_{3,t-1} + 0.13\lambda_{4,t-1} - 0.02a_{t-1} - 0.05u_t \\
\lambda_{4,t} &= -0.01\lambda_{1t-1} + 0.05\lambda_{3,t-1} + 0.62\lambda_{4,t-1} + 0.02a_{t-1} + 0.05u_t \\
\lambda_{5,t} &= 0.015\lambda_{1t-1} - 0.05\lambda_{3,t-1} + 0.13\lambda_{4,t-1} - .02a_{t-1} - 0.05u_t
\end{aligned}
$$

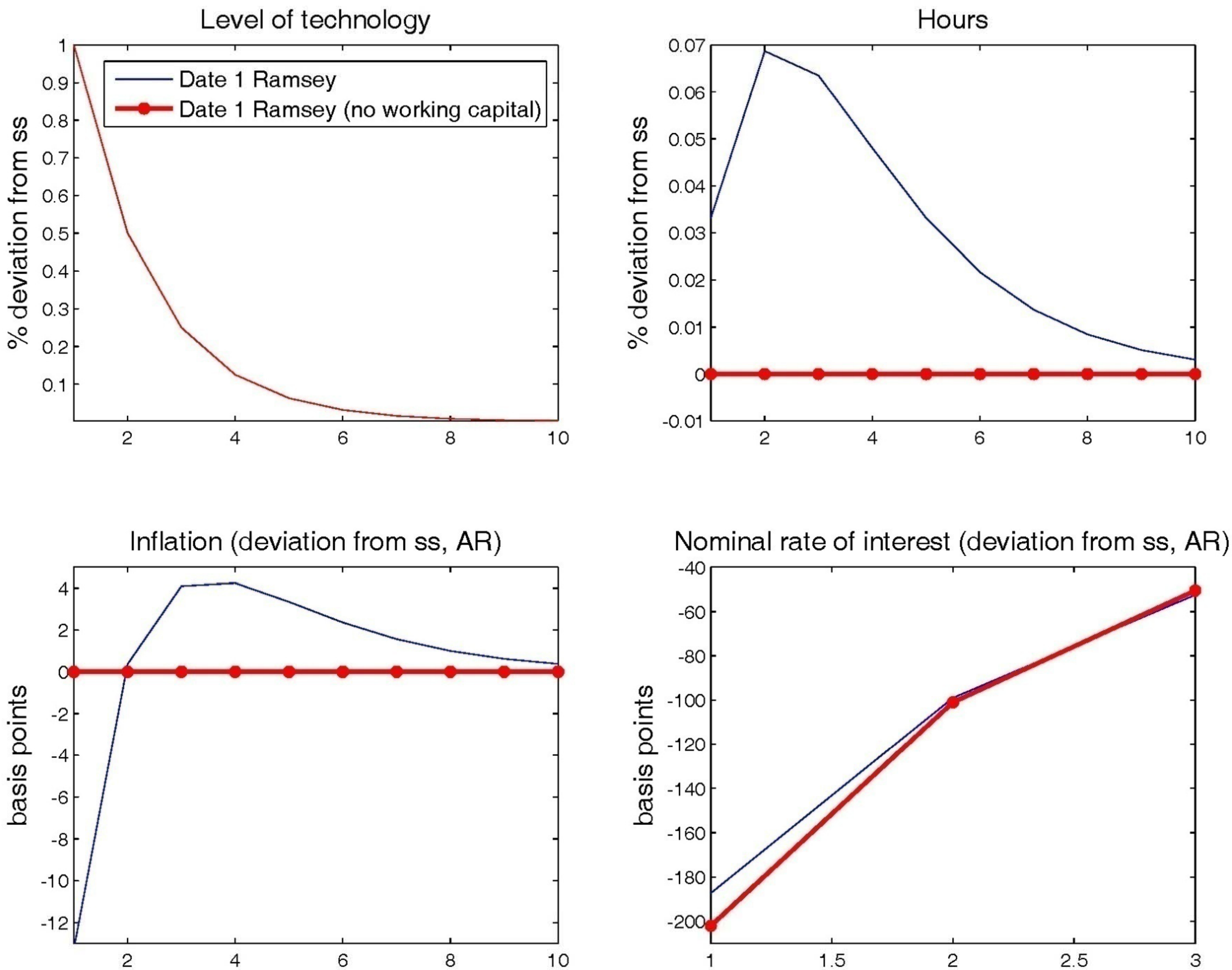
$$\lambda_1 = \lambda_3 = \lambda_4 = \lambda_5 = 0, \lambda_2 = 3.88$$

- Properties: all multipliers respond to u_t; optimal plan not time consistent; employment and inflation respond to u_t; r_t drops a little less than before (it's a tax now); N_t falls somewhat because of the interest rate 'tax'.

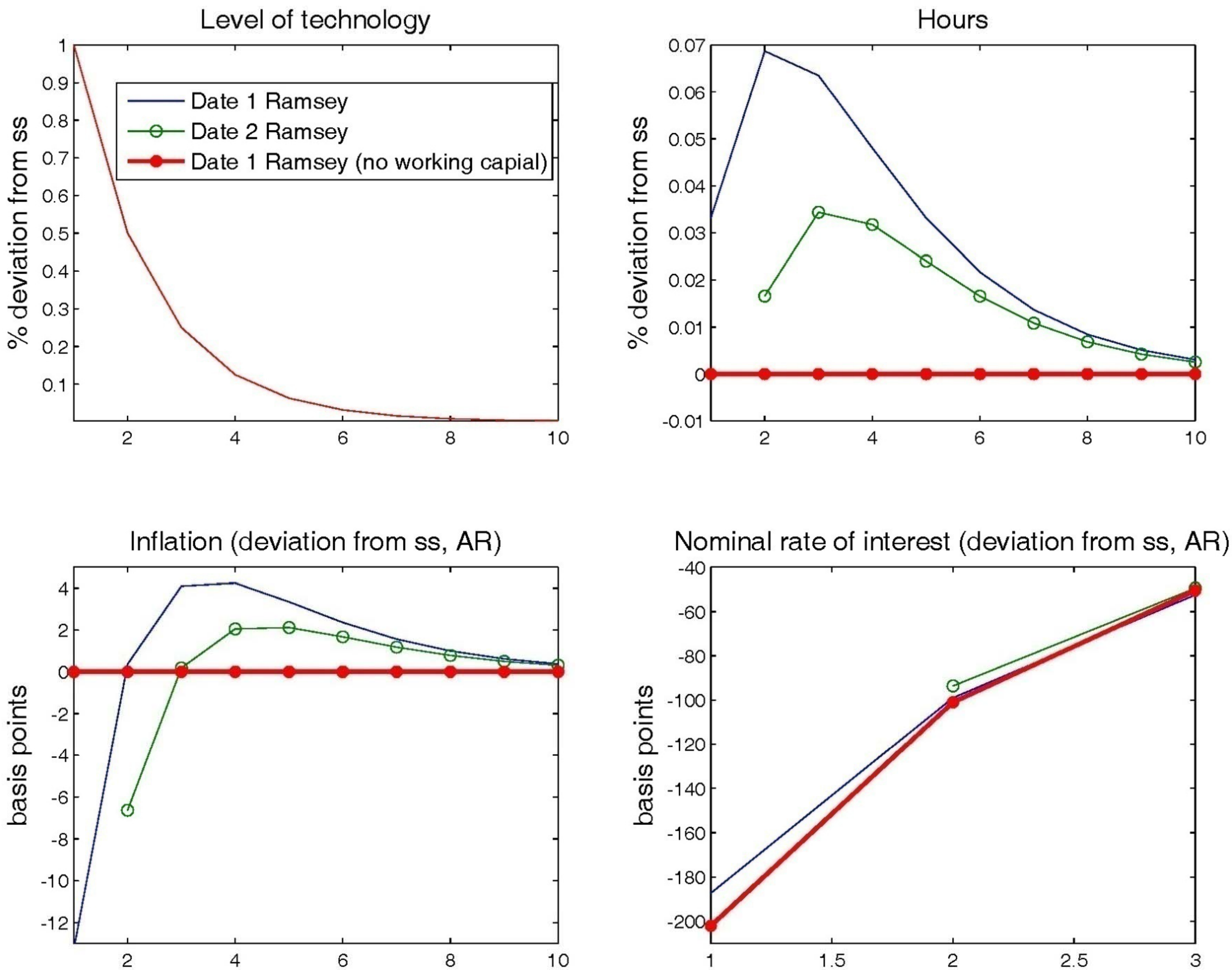
- Experiment:
  - Economy is in steady state of optimal plan up to period t.
  - A positive shock to technology occurs.
  - Monetary authority computes optimal policy and displays it in a set of charts.
  - Redo charts one period later.



Level of technology
% deviation from ss
Date 1 Ramsey (no working capital)
Hours
% deviation from ss
Inflation (deviation from ss, AR)
basis points
Nominal rate of interest (deviation from ss, AR)
basis points



Level of technology
% deviation from ss
Date 1 Ramsey
Date 1 Ramsey (no working capital)
Hours
% deviation from ss
Inflation (deviation from ss, AR)
basis points
Nominal rate of interest (deviation from ss, AR)
basis points



Level of technology
% deviation from ss
Date 1 Ramsey
Date 2 Ramsey
Date 1 Ramsey (no working capial)
Hours
% deviation from ss
Inflation (deviation from ss, AR)
basis points
Nominal rate of interest (deviation from ss, AR)
basis points

- Discussion of the results
  - In the absence of a working capital channel (i.e., $\psi = 0$) it is optimal to cut the interest rate, to encourage households not smooth consumption away from what is optimal.
  - In the presence of a working capital channel, (i.e., $\psi > 0$), the cut in the interest rate reduces the marginal cost of labor and expands output and employment. By reducing marginal cost, inflation drops.
  - The rise in employment and fall in inflation are both costly, and so:
    - it is optimal when $\psi > 0$ to cut the interest rate by less.
    - it is optimal to manage expectations so that the incentive to cut prices in the present is reduced.
      - announce inflation close to zero in the next period
      - announce relatively small interest rate drop in the next period.