# Technical Appendix to Risk, Return, and Dividends

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# A Relation of Proposition 2.1 to Pricing Kernel Formulations

By definition, given the dividend process $D_t$, the price of the stock is given by:

$$P_t = \mathrm{E}_t \left[ \int_t^\infty \Lambda_s D_s \, ds \right], \tag{A-1}$$

under the pricing kernel process $\Lambda_t$, together with a transversality assumption. We assume that the pricing kernel follows:

$$\frac{d\Lambda_t}{\Lambda_t} = -r_f(x_t)dt - \xi_x(x_t)dB_t^x - \xi_d(x_t)dB_t^d, \tag{A-2}$$

where $r_f(\cdot)$ is the risk-free rate process, and $\xi_x$ and $\xi_d$ are prices of risk corresponding to shocks to the state variable $x_t$ and dividend growth, respectively. Using equation (A-1), we can express the price-dividend ratio as:

$$\begin{aligned} \frac{P_t}{D_t} = \mathrm{E}_t \Bigg[ \int_t^\infty \exp\left( - \int_t^s (r_f + \frac{1}{2}(\xi_x^2 + \xi_d^2)) \, du + \xi_x \, dB_u^x + \xi_d \, dB_u^d \right) & \\ \times \exp\left( \int_t^s \mu_d du + \sigma_d dB_u^d \right) ds \Bigg], & \end{aligned}$$

assuming that $\sigma_{dx} = 0$ for simplicity. This can be equivalently written as:

$$\frac{P_t}{D_t} = \mathrm{E}_t^Q \left[ \int_t^\infty \exp\left( - \int_t^s (r_f - \mu_d - \frac{1}{2}(\sigma_d - \xi_d)^2) \, du \right) ds \right], \tag{A-3}$$

where the Radon-Nikodym derivative defining the risk-neutral measure $Q$ is given by:

$$\frac{dQ}{dP} = \exp\left( - \int_t^s \frac{1}{2}(\xi_x^2 + (\sigma_d - \xi_d)^2) \, du - \xi_x \, dB_u^x - (\sigma_d - \xi_d) dB_u^d \right). \tag{A-4}$$

Note that equation (A-3) is a function $f(\cdot)$ of $x$.

We describe how a particular choice of a return process $dR_t$, together with assumptions on dividends, places restrictions on the underlying pricing kernel process $d\Lambda_t$ through the following proposition:

**Proposition A.1** *Suppose the state of the economy is described by $x_t$, which follows equation (1), and a stock is a claim to the dividends $D_t$ that are described by equation (2) with $\sigma_{dx} = 0$. If the stock return follows equation (8) and the pricing kernel process follows equation (A-2), then the price-dividend ratio $P_t/D_t = f(x_t)$ satisfies the following relation:*

$$(\mu_x - \xi_x \sigma_x) f' + \frac{1}{2}\sigma_x^2 f'' - (r_f - \mu_d - \frac{1}{2}\sigma_d^2 + \xi_d \sigma_d) f = -1, \tag{A-5}$$

*which determines the price-dividend ratio $f$. This implies that the expected return $\mu_r(\cdot)$ and volatility $\sigma_{rx}(\cdot)$ of the return are given by:*

$$
\begin{aligned}
\mu_r &= r_f + \xi_x \sigma_x (\ln f)' + \xi_d \sigma_d, \\
\sigma_{rx} &= \sigma_x (\ln f)'
\end{aligned}
\tag{A-6}
$$

Proof: Equation (A-5) is the standard Feynman-Kac pricing equation. Once the price-dividend ratio $f$ is obtained from solving equation (A-5), we can derive equation (A-6) by equating terms from the drift term of $dR_t$ and the diffusion term on $dB_t^x$ in equation (8). ■

Proposition A.1 states that, given the dividend stream, the pricing kernel completely determines the price-dividend ratio $f$, the expected return of the stock $\mu_r$, and the volatility of the stock $\sigma_{rx}$. However, if we specify the price of the stock, the expected return, or the volatility of the stock (each one being sufficient to determine the other two from Proposition 2.1), the short rate $r^f$, the prices of risk $\xi_x$ and $\xi_d$, or the pricing kernel $\Lambda_t$ are not uniquely determined. For example, suppose we specify $\mu_r$. There are potentially infinitely many pairs of $r^f$ and $\xi = (\xi_x, \, \xi_d)$ that can produce the same $\mu_r$. For example, one (trivial) choice of $\xi$ is $\xi = (0, \, 0)$ corresponding to risk neutrality, and the stock return is the same as the risk-free rate. Whereas Proposition 2.1 shows that specifying $\mu_r$, $\sigma_{rx}$, or $f$ completely determines the return process, the result from Proposition A.1 implies that a single choice of $\mu_r$, $\sigma_{rx}$, or $f$ does not necessarily determine the pricing kernel.

# B Multivariate State Variables

Suppose that there are $K$ state variables, so that $x = (x_1, \ldots, x_K)^\top$ represents a $K \times 1$ vector of diffusion processes. We let $x$ follow the diffusion process in equation (1), where $\mu_x(\cdot)$ is a vector function of $x$ and $\sigma_x(\cdot)$ is a matrix function of $x$. Similarly, dividend growth satisfies equation (2) where the scalars $\mu_d(\cdot)$, $\sigma_d(\cdot)$ are potentially functions of $x$. For expositional simplicity, we assume that $\sigma_{dx} = 0$ and denote the scalar price-dividend ratio by $P/D = f(x)$.

Suppose that the return $R_t$ satisfies the following diffusion equation:

$$
\begin{aligned}
dR_t &= \mu_r(x_t)dt + \sigma_{rx}(x_t)dB_t^x + \sigma_{rd}(x_t)dB_t^d \\
&= \mu_r(x_t)dt + \sum_{i=1}^{K} \sigma_{rx_i}(x_t)dB_t^{x_i} + \sigma_{rd}(x_t)dB_t^d,
\end{aligned}
\tag{B-1}
$$

where $\mu_r(\cdot)$ is a scalar function of $x$, $\sigma_{rx}(\cdot)$ is a matrix function of $x$, $\sigma_{rx_i}$ represents the $i$th row of $\sigma_{rx}$, and the vector of Brownian motions $dB_t^x$ is partitioned as $dB_t^x = (dB_t^{x_1} \dots dB_t^{x_K})^\top$. From the definition of the return $dR_t = df_t/f_t + dD_t/D_t + 1/f_t dt$, the diffusion term of the return is given by:

$$\left(\frac{\partial \ln f}{\partial x}\right)^\top \sigma_{rx} dB_t^x + \sigma_d dB_t^d. \tag{B-2}$$

Thus, in order for $\sigma_{rx}$ to represent the diffusion coefficients of a return, we must have:

$$\sigma_{rx} = \left(\frac{\partial \ln f}{\partial x}\right)^\top \sigma_x, \tag{B-3}$$

or, equivalently:

$$\sigma_{rx_i} = \sum_{j=1}^{K} \frac{\partial \ln f}{\partial x_j} (\sigma_x)_{ji},$$

where $(\sigma_x)_{ji}$ is the element of $\sigma_x$ in the $j$th row and $i$th column. From equation (B-3), there must be a function $f$ such that:

$$\frac{\partial \ln f}{\partial x} = (\sigma_{rx}\sigma_x^{-1})^\top. \tag{B-4}$$

The necessary and sufficient condition for this is:

**Assumption B.1** *The diffusion coefficients $\sigma_x$ and $\sigma_{rx}$ satisfy the integrability condition:*

$$\frac{\partial}{\partial x_j}(\sigma_{rx}\sigma_x^{-1})_i = \frac{\partial}{\partial x_i}(\sigma_{rx}\sigma_x^{-1})_j. \tag{B-5}$$

Note that unlike the univariate case, we cannot arbitrarily specify the diffusion coefficients $\sigma_{rx}$ of the return. If $\sigma_{rx}$ does not satisfy the integrability condition, then equation (B-1) cannot represent a return implied from a pricing function. The multivariate version of equations (8) and (11) in Proposition 2.1 are:

**Proposition B.1** *Suppose that the return $R_t$ follows the diffusion equation (B-1) and that diffusion ratio $\sigma_{rx}\sigma_x^{-1}$ satisfies the integrability condition Assumption B.1. Then, the price-dividend ratio and the expected return are determined up to integration constant.*

Proof: From the integrability condition (B-5), it follows from elementary calculus that there exists a function $f$ that satisfies:

$$d\ln f = \sigma_{rx}\sigma_x^{-1} dx. \tag{B-6}$$

Equation (B-6) is the multivariate version of equation (11). The function $f$ is the dividend yield, and it is unique up to a multiplicative constant (since $\ln f$ is determined up to an additive constant). The expected return is then determined by:

$$\mu_r(x) = \frac{\mu_x^\top f' + \frac{1}{2} f''^\top \sigma_x \sigma_x^\top f'' + 1}{f} + \mu_d + \frac{1}{2}\sigma_d^2. \tag{B-7}$$

Equation (B-7) is the multivariate version of equation (8). ■

The integrability condition in Assumption B.1 imposes strong restrictions on multivariate stochastic volatility processes. For example, the diffusion process

$$\sqrt{v_1} dB_t^1 + v_1 \sqrt{v_2} dB_t^2,$$

where $v_1$ and $v_2$ are stochastic processes, cannot represent the return diffusion process of a valid pricing function because it violates the condition (B-5).