Technical Appendix for International Capital Flows under Dispersed Information: Theory and Evidence

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1 A two-country, one-good, two-asset model

1.1 Production and investment

There is one good produced in two countries, Home and Foreign. The production uses a constant returns to scale technology combining labor and capital:

$$Y_{i,t} = A_{i,t} K_{i,t}^{1-\omega} N_{i,t}^{\omega} \qquad\qquad i = H, F \tag{1}$$

where i denotes the specific country, with the Home and Foreign country denoted by $i = H$ and $i = F$ respectively. Y_i is the output in country i, A_i is an exogenous stochastic productivity term, K_i is the capital input and N_i the labor input. A share ω of output is paid to labor, with the remaining going to capital. For simplicity we fix the labor input to unity. The log of the productivity term follows an autoregressive process:

$$a_{i,t+1} = \rho a_{i,t} + \varepsilon_{i,t+1} \tag{2}$$

where $a_{i,t} = \ln(A_{i,t})$ and $\epsilon_{i,t+1}$ follows a $N(0, \sigma_a^2)$ distribution, uncorrelated across countries.

While the labor input is set, the capital stock can be adjusted:

$$K_{i,t+1} = (1-\delta) K_{i,t} + I_{i,t} \tag{3}$$

where δ is the depreciation rate and I_i is investment.

The capital is built by investment firms that transform consumption goods into capital goods at a cost. Consider an investment firm in the Home country. In period t it produces $I_{H,t}$ units of capital, which it sells them at a price $Q_{H,t}$, by using $I_{H,t}$ units of the consumption good. Production entails a cost, and the profits of the investment firm are:

$$Q_{H,t} I_{H,t} - I_{H,t} - \frac{\xi}{2} \frac{(I_{H,t} - \delta K_{H,t})^2}{K_{H,t}} \tag{4}$$

In (4) we assume that the individual firm takes the aggregate capital stock $K_{H,t}$ that enters the adjustment cost as given. We assume that the investment firm is a price taker, so the maximization of profits with respect to$I_{H,t}$ implies the standard Tobin's Q relation:

$$Q_{H,t} = 1 + \xi \frac{I_{H,t} - \delta K_{H,t}}{K_{H,t}} \tag{5}$$

The consumption good is produced by production firms who buy capital from the investment firms. The expected discounted profits of a Home production firm are:

$$
\begin{aligned}
& E_t^H \sum_{s=0}^{\infty} D_{H,t+s} \left[A_{H,t+s} K_{H,t+s}^{1-\omega} N_{H,t+s}^{\omega} - W_{t+s} N_{H,t+s} - Q_{H,t+s} I_{H,t+s} \right] \\
& - E_t^H \sum_{s=0}^{\infty} D_{H,t+s} \lambda_{H,t+s} \left[K_{H,t+1+s} - (1-\delta) K_{H,t+s} - I_{H,t+s} \right]
\end{aligned}
$$

where $D_{H,t+s}$ is the stochastic discount factor between time t and $t+s$. Recalling that the labor input is set to one, the first order conditions with respect to $K_{H,t+s+1}$ and $I_{H,t+s}$ are:

$$
\begin{aligned}
0 &= E_t^H D_{H,t+s+1} (1-\omega) A_{H,t+s+1} K_{H,t+s+1}^{-\omega} \\
& \quad - E_t^H D_{H,t+s} \lambda_{H,t+s} + (1-\delta) E_t^H D_{H,t+s+1} \lambda_{H,t+s+1} \\
0 &= -E_t^H D_{H,t+s} Q_{H,t+s} + E_t^H D_{H,t+s} \lambda_{H,t+s}
\end{aligned}
$$

Setting $s = 0$ we get $Q_{H,t} = \lambda_{H,t}$ and:

$$
\begin{aligned}
1 &= E_t^H \frac{D_{H,t+1}}{D_{H,t}} R_{H,t+1} \\
\exp[r_{H,t+1}] &= (1-\omega) \exp[a_{Ht+1} - \omega k_{H,t+1} - q_{H,t}] \\
& \quad + (1-\delta) \exp[q_{Ht+1} - q_{H,t}]
\end{aligned} \tag{6}
$$

where $R_{H,t+1}$ is the rate of return on Home capital, and lower case letters denote logs: $r_{H,t+1} = \ln[R_{H,t+1}]$. We get a similar expression for the rate of return on Foreign capital:

$$
\begin{aligned}
\exp[r_{F,t+1}] &= (1-\omega) \exp[a_{Ft+1} - \omega k_{F,t+1} - q_{F,t}] \\
& \quad + (1-\delta) \exp[q_{Ft+1} - q_{F,t}]
\end{aligned} \tag{7}
$$

The first order condition with respect to labor implies that $W_{H,t+s} = \omega A_{H,t+s} K_{H,t+s}^{1-\omega}$.

The dynamics of investment are driven by the Tobin's Q relation (5) and the capital accumulation relation (3):

$$
\exp[k_{H,t+1} - k_{H,t}] = 1 + \frac{\exp[q_{H,t}] - 1}{\xi} \tag{8}
$$

$$
\exp[k_{F,t+1} - k_{F,t}] = 1 + \frac{\exp[q_{F,t}] - 1}{\xi} \tag{9}
$$

1.2 Consumption and portfolio choice

We consider a two period overlapping generation framework. Agents work only when young and consume in both periods of their lives. They face two decisions: how much to save to fund their consumption, and in which assets to save. Agents in each country can invest in Home and Foreign equity (claims on capital) which yield the returns (6) and (7). While agents can invest in equity abroad, this entails a cost. Specifically, agent j in the Home country investing at time t in the Foreign country receives only the return at time $t+1$ times an iceberg cost $\exp[-\tau_t^{Hj}]$. Similarly, agent j in the Foreign country investing in the Home country receives the returns times an iceberg cost $\exp[-\tau_t^{Fj}]$. The transaction costs represent a simple way to capture the hurdles of investing abroad, reflecting the cost of gathering information on an unfamiliar market for instance.

The transaction costs can differ across various agents and across countries. Each agent j observes the cost on the future return at the time when she makes her portfolio choice. She can only observe her own cost of investing abroad, not the average cost faced by agents in her country. We denote the average cost faced by Home agents by τ_t^H, and the average cost faced by Foreign agents by τ_t^F. These costs are randomly distributed around a mean τ:

$$\left| \begin{array}{c} \left(\tau_t^H - \tau\right)/\tau \\ \left(\tau_t^F - \tau\right)/\tau \end{array} \right| \rightsquigarrow N\left(\left| \begin{array}{c} 0 \\ 0 \end{array} \right|, \theta\sigma_a^2 \left| \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right| \right) \tag{10}$$

where $\theta\sigma_a^2$ is the variance of Home and Foreign international transaction costs. We assume that the costs are perfectly negatively correlated, so the average transaction cost is constant (this is done for simplicity and does not alter our results). For convenience the variance is expressed relative to the variance of productivity innovations in (2). In period t a Home young agent j invests a fraction $z_{Hj,t}$ of her wealth in Home equity, and a fraction $1 - z_{Hj,t}$ in Foreign equity. The overall real return on her portfolio is then:

$$\exp\left[r_{t+1}^{p,Hj}\right] = z_{Hj,t} \exp\left[r_{H,t+1}\right] + (1 - z_{Hj,t}) \exp\left[r_{F,t+1} - \tau_t^{Hj}\right] \tag{11}$$

Similarly for the return on a Foreign agent's portfolio is:

$$\exp\left[r_{t+1}^{p,Fj}\right] = z_{Fj,t} \exp\left[r_{H,t+1} - \tau_t^{Fj}\right] + (1 - z_{Fj,t}) \exp\left[r_{F,t+1}\right] \tag{12}$$

We define the average portfolio shares of Home and Foreign investors as:

$$z_{H,t} = \int_0^1 z_{Hj,t} dj \qquad\qquad z_{F,t} = \int_0^1 z_{Hj,t} dj \tag{13}$$

A young Home agent j at time t makes consumption and portfolio decisions to maximize

$$\frac{\left(C_{y,t}^{Hj}\right)^{1-\gamma}}{1-\gamma}+\beta E_t^{Hj}\frac{\left(C_{o,t+1}^{Hj}\right)^{1-\gamma}}{1-\gamma}$$

where $C_{y,t}^{Hj}$ denotes the consumption when young at period t, and $C_{o,t+1}^{Hj}$ denotes consumption when old at period $t+1$. E_t^{Hj} denotes the expectations from the point of the view of the Home agent j. The income of a young agent is given by the wage $w_{H,t}=\omega A_{H,t}\left(K_{H,t}\right)^{1-\omega}$. The intertemporal budget constraint is then:

$$C_{o,t+1}^{Hj}=\left(\omega A_{Ht}\left(K_{H,t}\right)^{1-\omega}-C_{y,t}^{Hj}\right)R_{t+1}^{p,Hj} \tag{14}$$

The first-order conditions for consumption choice is the standard consumption Euler equation that links the ratio of marginal utilities of consumptions to the real interest rate:

$$\left(\omega\exp\left[a_{Ht}+(1-\omega)k_{H,t}-c_{y,t}^{Hj}\right]-1\right)^{\gamma}=\beta E_t^{Hj}\exp\left[(1-\gamma)r_{t+1}^{p,Hj}\right] \tag{15}$$

The first-order conditions for portfolio choice is the portfolio Euler equation that equalizes the expected discounted returns of all assets:

$$E_t^{Hj}\exp\left[r_{H,t+1}-\gamma r_{t+1}^{p,Hj}\right]=E_t^{Hj}\exp\left[r_{F,t+1}-\tau_t^{Hj}-\gamma r_{t+1}^{p,Hj}\right] \tag{16}$$

The corresponding relations for a Foreign agent are:

$$\left(\omega\exp\left[a_{Ft}+(1-\omega)k_{F,t}-c_{y,t}^{Fj}\right]-1\right)^{\gamma}=\beta E_t^{Fj}\exp\left[(1-\gamma)r_{t+1}^{p,Fj}\right] \tag{17}$$

and:

$$E_t^{Fj}\exp\left[r_{H,t+1}-\tau_t^{Fj}-\gamma r_{t+1}^{p,Fj}\right]=E_t^{Fj}\exp\left[r_{F,t+1}-\gamma r_{t+1}^{p,Fj}\right] \tag{18}$$

1.3 Private information

Agents receive private signals about productivity innovations in the next period. The signals received by a Home agent j about the log of Home and Foreign productivity innovations are

$$v_{j,t}^{H,H}=\varepsilon_{H,t+1}+\epsilon_{j,t}^{H,H}\qquad;\qquad v_{j,t}^{H,F}=\varepsilon_{F,t+1}+\epsilon_{j,t}^{H,F} \tag{19}$$

Each signal consists of the true innovation and an error specific to the agent. These errors are normally distributed with mean zero across agents in a given country.

The variance of errors across Home agents for Home and Foreign productivity are $\sigma^2_{H,H}$ and $\sigma^2_{H,F}$ respectively. Similarly, a Foreign agent j receives the signals

$$v^{F,H}_{j,t} = \varepsilon_{H,t+1} + \epsilon^{F,H}_{j,t} \qquad ; \qquad v^{F,F}_{j,t} = \varepsilon_{F,t+1} + \epsilon^{F,F}_{j,t} \tag{20}$$

The variance of errors across Foreign agents for Home and Foreign productivity are $\sigma^2_{H,F}$ and $\sigma^2_{H,H}$ respectively.

We assume that agents receive a better signal about domestic productivity: $\sigma^2_{H,H} < \sigma^2_{H,F}$.

1.4 Asset and goods market clearing

The clearing of the Home equity market requires that the value of Home capital at the end of the period matches the holdings of Home capital in investors portfolios:

$$\begin{aligned} & \exp\left[q_{Ht} + k_{H,t+1}\right] \\ = & \int \left(\omega \exp\left[a_{Ht} + (1-\omega)\, k_{H,t}\right] - \exp\left[c^{Hj}_{y,t}\right]\right) z_{Hj,t} dj \\ & + \int \left(\omega \exp\left[a_{Ft} + (1-\omega)\, k_{F,t}\right] - \exp\left[c^{Fj}_{y,t}\right]\right) z_{Fj,t} dj \end{aligned} \tag{21}$$

Similarly, the clearing of the Foreign equity market requires:

$$\begin{aligned} & \exp\left[q_{Ft} + k_{F,t+1}\right] \\ = & \int \left(\omega \exp\left[a_{Ht} + (1-\omega)\, k_{H,t}\right] - \exp\left[c^{Hj}_{y,t}\right]\right) (1 - z_{Hj,t})\, dj \\ & + \int \left(\omega \exp\left[a_{Ft} + (1-\omega)\, k_{F,t}\right] - \exp\left[c^{Fj}_{y,t}\right]\right) (1 - z_{Fj,t})\, dj \end{aligned} \tag{22}$$

Adding up (21) and (22), the value of worldwide capital matches world wealth:

$$\begin{aligned} & \exp\left[q_{Ht} + k_{H,t+1}\right] + \exp\left[q_{Ft} + k_{F,t+1}\right] \\ = & \ \omega \exp\left[a_{Ht} + (1-\omega)\, k_{H,t}\right] + \omega \exp\left[a_{Ft} + (1-\omega)\, k_{F,t}\right] \\ & - \int \exp\left[c^{Hj}_{y,t}\right] dj - \int \exp\left[c^{Fj}_{y,t}\right] dj \end{aligned}$$

The demand for the consumption good at time $t+1$ consists of the consumption of young agents in both countries, $C^{Hj}_{y,t+1}$ and $C^{Fj}_{y,t+1}$, the consumption of old agents in both countries, $C^{Hj}_{o,t+1}$ and $C^{Fj}_{o,t+1}$. We assume that the transaction cost on equity returns abroad is paid to brokers who fully consume it. Similarly the profits of investment firms are paid to the owners of these firms who fully consume their earnings. In addition to consumption, resources are used to transform consumption goods into capital. The various uses of output add up to world production:

$$\begin{aligned}
& \exp\left[y_{H,t+1}\right]+\exp\left[y_{F,t+1}\right] \\
= & \int \exp\left[c_{y,t+1}^{Hj}\right]dj+\int \exp\left[c_{y,t+1}^{Fj}\right]dj+\int \exp\left[c_{o,t+1}^{Hj}\right]dj+\int \exp\left[c_{o,t+1}^{Fj}\right]dj \\
& +\int\left(1-z_{Hj,t}\right)\left(\omega\exp\left[a_{Ht}+(1-\omega)k_{H,t}\right]-\exp\left[c_{y,t}^{Hj}\right]\right)\exp\left[r_{F,t+1}\right]\left(1-\exp\left[-\tau_t^{Hj}\right]\right)dj \\
& +\int z_{Fj,t}\left(\omega\exp\left[a_{Ft}+(1-\omega)k_{F,t}\right]-\exp\left[c_{y,t}^{Fj}\right]\right)\exp\left[r_{H,t+1}\right]\left(1-\exp\left[-\tau_t^{Fj}\right]\right)dj \\
& +\xi\frac{I_{H,t+1}-\delta K_{H,t+1}}{K_{H,t+1}}\left[I_{H,t+1}-\frac{1}{2}\left(I_{H,t+1}-\delta K_{H,t+1}\right)\right] \\
& +\xi\frac{I_{F,t+1}-\delta K_{F,t+1}}{K_{F,t+1}}\left[I_{F,t+1}-\frac{1}{2}\left(I_{F,t+1}-\delta K_{F,t+1}\right)\right] \\
& +I_{H,t+1}+\frac{\xi}{2}\frac{\left(I_{H,t+1}-\delta K_{H,t+1}\right)^2}{K_{H,t+1}}+I_{F,t+1}+\frac{\xi}{2}\frac{\left(I_{F,t+1}-\delta K_{F,t+1}\right)^2}{K_{F,t+1}}
\end{aligned} \tag{23}$$

(23) is redundant. Using the output (1), the consumer budget constraint (14), the Tobin's Q relation (5), the portfolio rates of returns (11)-(12) and the asset market clearing conditions (21)-(22), (23) becomes:

$$\begin{aligned}
& \exp\left[a_{Ht+1}+(1-\omega)k_{H,t+1}\right]+\exp\left[a_{Ft+1}+(1-\omega)k_{F,t+1}\right] \\
= & \int \exp\left[c_{y,t+1}^{Hj}\right]dj+\int \exp\left[c_{y,t+1}^{Fj}\right]dj \\
& +\exp\left[q_{Ht}+k_{H,t+1}\right]\exp\left[r_{H,t+1}\right]+\exp\left[q_{Ft}+k_{F,t+1}\right]\exp\left[r_{F,t+1}\right] \\
& +\exp\left[q_{H,t+1}\right]\left(\exp\left[k_{H,t+2}\right]-(1-\delta)\exp\left[k_{H,t+1}\right]\right) \\
& +\exp\left[q_{F,t+1}\right]\left(\exp\left[k_{F,t+2}\right]-(1-\delta)\exp\left[k_{F,t+1}\right]\right)
\end{aligned}$$

Using the return on Home and Foreign equity (6)-(7), and the asset market clearing conditions (21)-(22) at time $t+1$ shows that this relation is redundant.

A useful definition is the worldwide average of portfolio shares, and their cross-country differences:

$$\begin{aligned}
z_{H,t} &= \int z_{Hj,t}dj \quad ; \quad z_{F,t}=\int z_{Fj,t}dj \\
z_t^A &= \frac{1}{2}\left[z_{H,t}+z_{F,t}\right] \quad ; \quad z_t^D=z_{H,t}-z_{F,t}
\end{aligned} \tag{24}$$

2 Expansion of the model

2.1 Steady state

The model is expanded around a zero-order allocation where productivity is normalized to unity: $a_F(0) = a_F(0) = 0$. (8) and (9) imply that the asset prices are unity: $q_H(0) = q_F(0) = 0$.

As the transaction cost τ is second order, (6)-(7) and (11)-(12) imply that all zero-order rates of return are:

$$\exp[r(0)] = (1-\omega)\exp[-\omega k(0)] + (1-\delta)$$

The consumption Euler relations (15) and (17) imply:

$$(\omega \exp[(1-\omega)k(0) - c_y(0)] - 1)^{\gamma} = \beta \exp[(1-\gamma)r(0)]$$

The sum of the asset market clearing conditions (21)-(22) implies:

$$\exp[k(0)] = \omega \exp[(1-\omega)k(0)] - \exp[c_y(0)]$$

These three relations lead to a non-linear solution for the steady state capital:

$$(\omega \exp[-\omega k(0)] - 1)^{-\gamma} = \beta[(1-\omega)\exp[-\omega k(0)] + (1-\delta)]^{1-\gamma} \qquad (25)$$

We define the propensity to consume of young agents as:

$$\bar{c} = \frac{\exp[c_y(0)]}{\exp[w(0)]} = \frac{\exp[c_y(0)]}{\omega \exp[(1-\omega)k(0)]}$$

Notice that the heterogeneity in zero-order portfolio choice does not lead to any heterogeneity in zero-order consumption, as all young agents face the same zero-order wage and returns.

The clearing of the Home equity market (21) gives the value of the average portfolio share (24):

$$z^A(0) = 0.5$$

2.2 Return on equity

We now write approximations around the zero-order allocation outlined above. We take cubic approximation of the log of (6):

$$\begin{aligned} r_{H,t+1} &= \ln\left[(1-\omega)\exp\left[a_{Ht+1} - \omega k_{H,t+1} - q_{H,t}\right] + (1-\delta)\exp\left[q_{Ht+1} - q_{H,t}\right]\right] \\ &= r(0) + (1-r_q)\left[a_{Ht+1} - \omega k_{H,t+1}\right] + r_q q_{Ht+1} - q_{H,t} \\ &\quad + \frac{r_q(1-r_q)}{2}\left[q_{Ht+1} - a_{Ht+1} + \omega k_{H,t+1}\right]^2 \\ &\quad + \frac{1-2r_q}{6} r_q (1-r_q)\left[q_{Ht+1} - a_{Ht+1} + \omega k_{H,t+1}\right]^3 \end{aligned} \tag{26}$$

where:

$$r_q = \frac{1-\delta}{(1-\omega)\exp\left[-\omega k(0)\right] + (1-\delta)}$$

Similarly (7) is expanded as:

$$\begin{aligned} r_{F,t+1} &= r(0) + (1-r_q)\left[a_{Ft+1} - \omega k_{F,t+1}\right] + r_q q_{Ft+1} - q_{F,t} \\ &\quad + \frac{r_q(1-r_q)}{2}\left[q_{Ft+1} - a_{Ft+1} + \omega k_{F,t+1}\right]^2 \\ &\quad + \frac{1-2r_q}{6} r_q (1-r_q)\left[q_{Ft+1} - a_{Ft+1} + \omega k_{F,t+1}\right]^3 \end{aligned} \tag{27}$$

For a variable x we define the worldwide average x^A and cross country difference x^A as:

$$x^A = 0.5\left(x_H + x_F\right) \qquad ; \qquad x^D = x_H - x_F \tag{28}$$

$$\Rightarrow x_H = x^A + 0.5x^D \qquad ; \qquad x_F = x^A - 0.5x^D$$

(26)-(27) can then be written as:

$$
\begin{aligned}
r_{H,t+1} &= r(0) + \left[(1-r_q)\left[a^A_{t+1} - \omega k^A_{t+1}\right] + r_q q^A_{t+1} - q^A_t\right] \\
&+\frac{1}{2}\left[(1-r_q)\left[a^D_{t+1} - \omega k^D_{t+1}\right] + r_q q^D_{t+1} - q^D_t\right] \\
&+\frac{r_q(1-r_q)}{2}\left[\left[q^A_{t+1} - a^A_{t+1} + \omega k^A_{t+1}\right] + \frac{1}{2}\left[q^D_{t+1} - a^D_{t+1} + \omega k^D_{t+1}\right]\right]^2 \\
&+\frac{1-2r_q}{6} r_q (1-r_q)\left[\left[q^A_{t+1} - a^A_{t+1} + \omega k^A_{t+1}\right] + \frac{1}{2}\left[q^D_{t+1} - a^D_{t+1} + \omega k^D_{t+1}\right]\right]^3 \\
r_{F,t+1} &= r(0) + \left[(1-r_q)\left[a^A_{t+1} - \omega k^A_{t+1}\right] + r_q q^A_{t+1} - q^A_t\right] \\
&-\frac{1}{2}\left[(1-r_q)\left[a^D_{t+1} - \omega k^D_{t+1}\right] + r_q q^D_{t+1} - q^D_t\right] \\
&+\frac{r_q(1-r_q)}{2}\left[\left[q^A_{t+1} - a^A_{t+1} + \omega k^A_{t+1}\right] - \frac{1}{2}\left[q^D_{t+1} - a^D_{t+1} + \omega k^D_{t+1}\right]\right]^2 \\
&+\frac{1-2r_q}{6} r_q (1-r_q)\left[\left[q^A_{t+1} - a^A_{t+1} + \omega k^A_{t+1}\right] - \frac{1}{2}\left[q^D_{t+1} - a^D_{t+1} + \omega k^D_{t+1}\right]\right]^3
\end{aligned}
$$

Taking the average of these relations, up to the quadratic terms, we write the average rate of return as:

$$
\begin{aligned}
r^A_{t+1} &= 0.5\left(r_{H,t+1} + r_{F,t+1}\right) \\
&= r(0) + \left[(1-r_q)\left[a^A_{t+1} - \omega k^A_{t+1}\right] + r_q q^A_{t+1} - q^A_t\right] \qquad (29) \\
&+\frac{r_q(1-r_q)}{2}\left[\left[q^A_{t+1} - a^A_{t+1} + \omega k^A_{t+1}\right]^2 + \frac{1}{4}\left[q^D_{t+1} - a^D_{t+1} + \omega k^D_{t+1}\right]^2\right]
\end{aligned}
$$

Taking the difference of these relations, up to the quadratic terms, we write the excess return on Home equity as:

$$
\begin{aligned}
er_{t+1} &= r_{H,t+1} - r_{F,t+1} \\
&= \left[(1-r_q)\left[a^D_{t+1} - \omega k^D_{t+1}\right] + r_q q^D_{t+1} - q^D_t\right] \qquad (30) \\
&+ r_q(1-r_q)\left[q^A_{t+1} - a^A_{t+1} + \omega k^A_{t+1}\right]\left[q^D_{t+1} - a^D_{t+1} + \omega k^D_{t+1}\right]
\end{aligned}
$$

2.3 Portfolio return

Next, we write quadratic approximations of the portfolio returns. (11) is written as (recalling that the friction τ_t^{Hj} is order two and above):

$$
\begin{aligned}
r_{t+1}^{p,Hj} &= \ln\left[z_{Hj,t}\exp\left[r_{H,t+1}\right] + (1 - z_{Hj,t})\exp\left[r_{F,t+1} - \tau_t^{Hj}\right]\right] \\
&= r(0) + z_{Hj}(0)\, r_{H,t+1} + (1 - z_{Hj}(0))\left(r_{F,t+1} - \tau_t^{Hj}\right) \\
&\quad + \frac{z_{Hj}(0)(1 - z_{Hj}(0))}{2}\left[er_{t+1}\right]^2 + z_{Hj,t}er_{t+1}
\end{aligned} \tag{31}
$$

where $er_{t+1} = r_{H,t+1} - r_{F,t+1}$. Similarly, (12) is written as:

$$
\begin{aligned}
r_{t+1}^{p,Fj} &= r(0) + z_{Fj}(0)\left(r_{H,t+1} - \tau_t^{Fj}\right) + (1 - z_{Fj}(0))\, r_{F,t+1} \\
&\quad + \frac{z_{Fj}(0)(1 - z_{Fj}(0))}{2}\left[er_{t+1}\right]^2 + z_{Fj,t}er_{t+1}
\end{aligned} \tag{32}
$$

2.4 Investment and asset market clearing

A quadratic approximation of (8) and (9) yields:

$$
\begin{aligned}
k_{H,t+1} - k_{H,t} &= \ln\left(1 + \frac{\exp\left[q_{H,t}\right] - 1}{\xi}\right) \\
&= \frac{1}{\xi}q_{H,t} + \frac{1}{2}\frac{\xi - 1}{\xi^2}\left(q_{H,t}\right)^2 \\
k_{F,t+1} - k_{F,t} &= \frac{1}{\xi}q_{F,t} + \frac{1}{2}\frac{\xi - 1}{\xi^2}\left(q_{F,t}\right)^2
\end{aligned}
$$

Using (28) we take the worldwide average of these relations, as well as their difference, to write:

$$
k_{t+1}^A - k_t^A = \frac{1}{\xi}q_t^A + \frac{1}{2}\frac{\xi - 1}{\xi^2}\left[\left(q_t^A\right)^2 + \frac{1}{4}\left(q_t^D\right)^2\right] \tag{33}
$$

$$
k_{t+1}^D - k_t^D = \frac{1}{\xi}q_t^D + \frac{\xi - 1}{\xi^2}q_t^A q_t^D \tag{34}
$$

The sum of the market clearing conditions (21)-(22) is expanded as:

$$\begin{aligned}
& q_t^A + k_{t+1}^A + \frac{1}{2}\left[\left(q_t^A + k_{t+1}^A\right)^2 + \frac{1}{4}\left(q_t^D + k_{t+1}^D\right)^2\right] \\
= & \frac{1}{1-\bar{c}}\left[a_t^A + (1-\omega)\, k_t^A\right] - \frac{\bar{c}}{1-\bar{c}} c_{yt}^A \\
& + \frac{1}{1-\bar{c}}\frac{1}{2}\left[\left(a_t^A + (1-\omega)\, k_t^A\right)^2 + \frac{1}{4}\left(a_t^D + (1-\omega)\, k_t^D\right)^2\right] \\
& - \frac{\bar{c}}{1-\bar{c}}\frac{1}{2}\left[\left(c_{yt}^A\right)^2 + \frac{1}{4}\left(c_{yt}^D\right)^2\right] - \frac{\bar{c}}{1-\bar{c}}\frac{1}{4}\left[D_t^H(c) + D_t^F(c)\right]
\end{aligned} \tag{35}$$

where:

$$\begin{aligned}
c_{y,t}^H &= \int c_{y,t}^{Hj} dj \quad ; \quad c_{yt}^F = \int c_{y,t}^{Fj} dj \\
c_{yt}^A &= \frac{1}{2}\left[c_{y,t}^H + c_{yt}^F\right] \quad ; \quad c_{yt}^D = c_{y,t}^H - c_{yt}^F \\
D_t^H(c) &= \int \left(c_{yt}^{Hj}\right)^2 dj - \left(c_{y,t}^H\right)^2 \quad ; \quad D_t^F(c) = \int \left(c_{yt}^{Fj}\right)^2 dj - \left(c_{yt}^F\right)^2
\end{aligned}$$

A linear approximation of the difference between the market clearing conditions (21) and (22) leads to:

$$\begin{aligned}
q_t^D + k_{t+1}^D = & \int \left(\frac{1}{1-\bar{c}}\left[a_{Ht} + (1-\omega)\, k_{H,t}\right] - \frac{\bar{c}}{1-\bar{c}} c_{y,t}^{Hj}\right)\left(2z_{Hj,t}(0) - 1\right) dj \\
& + \int \left(\frac{1}{1-\bar{c}}\left[a_{Ft} + (1-\omega)\, k_{F,t}\right] - \frac{\bar{c}}{1-\bar{c}} c_{y,t}^{Fj}\right)\left(2z_{Fj,t}(0) - 1\right) dj \\
& + 4z_t^A
\end{aligned} \tag{36}$$

where:

$$z_t^A = 0.5\left[\int z_{Hj,t} dj + \int z_{Fj,t} dj\right]$$

2.5 Consumption Euler equations

A quadratic approximation of (15) yields:

$$\begin{aligned}
& \frac{\gamma}{1-\bar{c}}\left[a_{Ht} + (1-\omega)\, k_{H,t} - c_{y,t}^{Hj}\right] + \frac{\gamma(\gamma-\bar{c})}{2(1-\bar{c})^2}\left[a_{Ht} + (1-\omega)\, k_{H,t} - c_{y,t}^{Hj}\right]^2 \\
= & E_t^{Hj}\left[(1-\gamma)\, r_{t+1}^{p,Hj} + \frac{1}{2}(1-\gamma)^2\left[r_{t+1}^{p,Hj}\right]^2\right]
\end{aligned}$$

Similarly (17) is approximated as:

$$
\begin{aligned}
& \frac{\gamma}{1-\bar{c}}\left[a_{Ft}+(1-\omega)\,k_{F,t}-c_{y,t}^{Fj}\right]+\frac{\gamma\,(\gamma-\bar{c})}{2\,(1-\bar{c})^{2}}\left[a_{Ft}+(1-\omega)\,k_{F,t}-c_{y,t}^{Fj}\right]^{2} \\
= \; & E_{t}^{Fj}\left[(1-\gamma)\,r_{t+1}^{p,Fj}+\frac{1}{2}\,(1-\gamma)^{2}\left[r_{t+1}^{p,Fj}\right]^{2}\right]
\end{aligned}
$$

We average these relations across all agents in each country. Using (31)-(32) and dropping terms of order three and above, we write:

$$
\begin{aligned}
& \frac{\gamma}{1-\bar{c}}\left[\left[a_{t}^{A}+(1-\omega)k_{t}^{A}\right]+\frac{1}{2}\left[a_{t}^{D}+(1-\omega)k_{t}^{D}\right]-\left[c_{yt}^{A}+\frac{1}{2}c_{yt}^{D}\right]\right] \\
& +\frac{\gamma\,(\gamma-\bar{c})}{2\,(1-\bar{c})^{2}}\left[\begin{array}{c}\left[\left[a_{t}^{A}+(1-\omega)k_{t}^{A}\right]+\frac{1}{2}\left[a_{t}^{D}+(1-\omega)k_{t}^{D}\right]-\left(c_{yt}^{A}+\frac{1}{2}c_{yt}^{D}\right)\right]^{2} \\ +D_{t}^{H}\,(c)\end{array}\right] \\
= \; & (1-\gamma)\int\left[\begin{array}{c}E_{t}^{Hj}r_{t+1}^{A}+\frac{2z_{Hj}(0)-1}{2}E_{t}^{Hj}er_{t+1}-(1-z_{Hj}\,(0))\,\tau \\ +\frac{z_{Hj}(0)\left(1-z_{Hj}(0)\right)}{2}E_{t}^{Hj}\left[er_{t+1}\right]^{2}+z_{Hj,t}E_{t}^{Hj}er_{t+1}\end{array}\right]dj \\
& +\frac{(1-\gamma)^{2}}{2}\int E_{t}^{Hj}\left[r_{t+1}^{A}+\frac{2z_{Hj}\,(0)-1}{2}er_{t+1}\right]^{2}dj
\end{aligned}
$$

where we used the fact that the second order component of τ_{t}^{Hj} is the same for all. Similarly for the Foreign country:

$$
\begin{aligned}
& \frac{\gamma}{1-\bar{c}}\left[\left[a_{t}^{A}+(1-\omega)k_{t}^{A}\right]-\frac{1}{2}\left[a_{t}^{D}+(1-\omega)k_{t}^{D}\right]-\left[c_{yt}^{A}-\frac{1}{2}c_{yt}^{D}\right]\right] \\
& +\frac{\gamma\,(\gamma-\bar{c})}{2\,(1-\bar{c})^{2}}\left[\begin{array}{c}\left[\left[a_{t}^{A}+(1-\omega)k_{t}^{A}\right]-\frac{1}{2}\left[a_{t}^{D}+(1-\omega)k_{t}^{D}\right]-\left(c_{yt}^{A}-\frac{1}{2}c_{yt}^{D}\right)\right]^{2} \\ +D_{t}^{F}\,(c)\end{array}\right] \\
= \; & (1-\gamma)\int\left[\begin{array}{c}E_{t}^{Fj}r_{t+1}^{A}+\frac{2z_{Fj}(0)-1}{2}E_{t}^{Fj}er_{t+1}-z_{Fj}\,(0)\,\tau \\ +\frac{z_{Fj}(0)\left(1-z_{Fj}(0)\right)}{2}E_{t}^{Fj}\left[er_{t+1}\right]^{2}+z_{Fj,t}E_{t}^{Fj}er_{t+1}\end{array}\right]dj \\
& +\frac{(1-\gamma)^{2}}{2}\int E_{t}^{Fj}\left[r_{t+1}^{A}+\frac{2z_{Fj}\,(0)-1}{2}er_{t+1}\right]^{2}dj
\end{aligned}
$$

Taking the worldwide average of these relations, we write:

$$
\begin{aligned}
& \frac{\gamma}{1-\bar{c}}\left[a_t^A+(1-\omega)k_t^A-c_{yt}^A\right] \\
& +\frac{\gamma\left(\gamma-\bar{c}\right)}{4\left(1-\bar{c}\right)^2}\left[2\left[a_t^A+(1-\omega)k_t^A-c_{yt}^A\right]^2+\frac{1}{2}\left[a_t^D+(1-\omega)k_t^D-c_{yt}^D\right]^2\right] \\
& +\frac{\gamma\left(\gamma-\bar{c}\right)}{4\left(1-\bar{c}\right)^2}\left[D_t^H\left(c\right)+D_t^F\left(c\right)\right] \\
= & \ \frac{1-\gamma}{2}\left[\begin{array}{c} \int E_t^{Hj}\left[\begin{array}{c} r_{t+1}^A+\frac{2z_{Hj}(0)-1}{2}er_{t+1} \\ +\frac{z_{Hj}(0)\left(1-z_{Hj}(0)\right)}{2}\left[er_{t+1}\right]^2+z_{Hj,t}er_{t+1}\end{array}\right]dj \\ +\int E_t^{Fj}\left[\begin{array}{c} r_{t+1}^A+\frac{2z_{Fj}(0)-1}{2}er_{t+1} \\ +\frac{z_{Fj}(0)\left(1-z_{Fj}(0)\right)}{2}\left[er_{t+1}\right]^2+z_{Fj,t}er_{t+1}\end{array}\right]dj \end{array}\right] \\
& -\frac{1-\gamma}{2}\left(1-z^D\left(0\right)\right)\tau \\
& +\frac{\left(1-\gamma\right)^2}{4}\left[\begin{array}{c} \int E_t^{Hj}\left[r_{t+1}^A+\frac{2z_{Hj}(0)-1}{2}er_{t+1}\right]^2 dj \\ +\int E_t^{Fj}\left[r_{t+1}^A+\frac{2z_{Fj}(0)-1}{2}er_{t+1}\right]^2 dj \end{array}\right]
\end{aligned} \tag{37}
$$

Taking the cross-country difference of these relations, we write:

$$
\begin{aligned}
& \frac{\gamma}{1-\bar{c}}\left[a_t^D+(1-\omega)k_t^D-c_{yt}^D\right] \\
& +\frac{\gamma\left(\gamma-\bar{c}\right)}{\left(1-\bar{c}\right)^2}\left[a_t^A+(1-\omega)k_t^A-c_{yt}^A\right]\left[a_t^D+(1-\omega)k_t^D-c_{yt}^D\right] \\
& +\frac{\gamma\left(\gamma-\bar{c}\right)}{2\left(1-\bar{c}\right)^2}\left[D_t^H\left(c\right)-D_t^F\left(c\right)\right] \\
= & \ (1-\gamma)\left[\begin{array}{c} \int E_t^{Hj}\left[\begin{array}{c} r_{t+1}^A+\frac{2z_{Hj}(0)-1}{2}er_{t+1} \\ +\frac{z_{Hj}(0)\left(1-z_{Hj}(0)\right)}{2}\left[er_{t+1}\right]^2+z_{Hj,t}er_{t+1}\end{array}\right]dj \\ -\int E_t^{Fj}\left[\begin{array}{c} r_{t+1}^A+\frac{2z_{Fj}(0)-1}{2}er_{t+1} \\ +\frac{z_{Fj}(0)\left(1-z_{Fj}(0)\right)}{2}\left[er_{t+1}\right]^2+z_{Fj,t}er_{t+1}\end{array}\right]dj \end{array}\right] \\
& +\frac{\left(1-\gamma\right)^2}{2}\left[\begin{array}{c} \int E_t^{Hj}\left[r_{t+1}^A+\frac{2z_{Hj}(0)-1}{2}er_{t+1}\right]^2 dj \\ -\int E_t^{Fj}\left[r_{t+1}^A+\frac{2z_{Fj}(0)-1}{2}er_{t+1}\right]^2 dj \end{array}\right]
\end{aligned} \tag{38}
$$

2.6 Portfolio Euler equations

2.6.1 General form

A cubic approximation of (16) yields:

$$\begin{aligned} 0 &= E_t^{Hj} er_{t+1} + \tau_t^{Hj} \\ &+ \frac{1}{2} E_t^{Hj} \left[\left[r_{t+1}^A + \frac{1}{2} er_{t+1} - \gamma r_{t+1}^{p,Hj} \right]^2 - \left[r_{t+1}^A - \frac{1}{2} er_{t+1} - \tau_t^{Hj} - \gamma r_{t+1}^{p,Hj} \right]^2 \right] \\ &+ \frac{1}{6} E_t^{Hj} \left[\left[r_{t+1}^A + \frac{1}{2} er_{t+1} - \gamma r_{t+1}^{p,Hj} \right]^3 - \left[r_{t+1}^A - \frac{1}{2} er_{t+1} - \gamma r_{t+1}^{p,Hj} \right]^3 \right] \end{aligned}$$

Similarly (18) is expanded as:

$$\begin{aligned} 0 &= E_t^{Fj} er_{t+1} - \tau_t^{Fj} \\ &+ \frac{1}{2} E_t^{Fj} \left[\left[r_{t+1}^A + \frac{1}{2} er_{t+1} - \tau_t^{Fj} - \gamma r_{t+1}^{p,Fj} \right]^2 - \left[r_{t+1}^A - \frac{1}{2} er_{t+1} - \gamma r_{t+1}^{p,Fj} \right]^2 \right] \\ &+ \frac{1}{6} E_t^{Fj} \left[\left[r_{t+1}^A + \frac{1}{2} er_{t+1} - \gamma r_{t+1}^{p,Fj} \right]^3 - \left[r_{t+1}^A - \frac{1}{2} er_{t+1} - \gamma r_{t+1}^{p,Fj} \right]^3 \right] \end{aligned}$$

Rearranging, these relation becomes:

$$\begin{aligned} 0 &= E_t^{Hj} er_{t+1} + \tau_t^{Hj} \\ &- \frac{1}{2} \tau_t^{Hj} E_t^{Hj} er_{t+1} + E_t^{Hj} \left(r_{t+1}^A - \gamma r_{t+1}^{p,Hj} \right) \left(er_{t+1} + \tau_t^{Hj} \right) \\ &+ \frac{1}{2} E_t^{Hj} \left(r_{t+1}^A - \gamma r_{t+1}^{p,Hj} \right)^2 er_{t+1} + \frac{1}{24} E_t^{Hj} \left[er_{t+1} \right]^3 \end{aligned} \tag{39}$$

and:

$$\begin{aligned} 0 &= E_t^{Fj} er_{t+1} - \tau_t^{Fj} \\ &- \frac{1}{2} \tau_t^{Fj} E_t^{Fj} er_{t+1} + E_t^{Fj} \left(r_{t+1}^A - \gamma r_{t+1}^{p,Fj} \right) \left(er_{t+1} - \tau_t^{Fj} \right) \\ &+ \frac{1}{2} E_t^{Fj} \left(r_{t+1}^A - \gamma r_{t+1}^{p,Fj} \right)^2 er_{t+1} + \frac{1}{24} E_t^{Fj} \left(er_{t+1} \right)^3 \end{aligned} \tag{40}$$

2.6.2 Second order terms

The first-order terms of (39)-(39) imply that first order expected returns are zero for all agents:

$$0 = \left[E_t^{Hj} er_{t+1} \right](1) = \left[E_t^{Fj} er_{t+1} \right](1) \tag{41}$$

Turning to the second order terms of (39)-(39), using the portfolio returns (31)-(32). We write:

$$\begin{aligned} 0 &= \left[E_t^{Hj} er_{t+1}\right](2) + \tau + (1-\gamma)\left[E_t^{Hj} r_{t+1}^A er_{t+1}\right](2) \\ &\quad -\gamma \frac{2z_{Hj}(0)-1}{2}\left[E_t^{Hj}(er_{t+1})^2\right](2) \\ 0 &= \left[E_t^{Fj} er_{t+1}\right](2) - \tau + (1-\gamma)\left[E_t^{Fj} r_{t+1}^A er_{t+1}\right](2) \\ &\quad -\gamma \frac{2z_{Fj}(0)-1}{2}\left[E_t^{Fj}(er_{t+1})^2\right](2) \end{aligned}$$

The zero-order portfolio share for a Home agent is then:

$$z_{Hj}(0) = \frac{1}{2} + \frac{\left[E_t^{Hj} er_{t+1}\right](2) + \tau}{\gamma\left[E_t^{Hj}(er_{t+1})^2\right](2)} + \frac{1-\gamma}{\gamma}\frac{\left[E_t^{Hj} r_{t+1}^A er_{t+1}\right](2)}{\left[E_t^{Hj}(er_{t+1})^2\right](2)} \tag{42}$$

Similarly for a Foreign agent:

$$z_{Fj}(0) = \frac{1}{2} + \frac{\left[E_t^{Fj} er_{t+1}\right](2) - \tau}{\gamma\left[E_t^{Fj}(er_{t+1})^2\right](2)} + \frac{1-\gamma}{\gamma}\frac{\left[E_t^{Fj} r_{t+1}^A er_{t+1}\right](2)}{\left[E_t^{Fj}(er_{t+1})^2\right](2)} \tag{43}$$

We show below that the second-order variance of the excess return is the same for all agents: $\left[E_t^{Hj}(er_{t+1})^2\right](2) = \left[E_t^{Fj}(er_{t+1})^2\right](2) = \left[E_t(er_{t+1})^2\right](2)$. For convenience we define the following measures of average expectations across Home agents and Foreign agents:

$$\bar{E}_t^H[x] = \int E_t^{Hj}[x]\,dj \qquad ; \qquad \bar{E}_t^F[x] = \int E_t^{Fj}[x]\,dj$$

Integrating (42)-(43) across agents in one country and combining we get, as $z^A(0) = 0.5$:

$$\begin{aligned} 0 &= \left[\bar{E}_t^H er_{t+1}\right](2) + (1-\gamma)\left[\bar{E}_t^H r_{t+1}^A er_{t+1}\right](2) \\ &\quad + \left[\bar{E}_t^F er_{t+1}\right](2) + (1-\gamma)\left[\bar{E}_t^F r_{t+1}^A er_{t+1}\right](2) \end{aligned} \tag{44}$$

and:

$$\begin{aligned} z^D(0) &= \frac{2\tau}{\gamma\left[E_t(er_{t+1})^2\right](2)} + \frac{\left[\bar{E}_t^H er_{t+1}\right](2) - \left[\bar{E}_t^F er_{t+1}\right](2)}{\gamma\left[E_t(er_{t+1})^2\right](2)} \\ &\quad + \frac{1-\gamma}{\gamma}\frac{\left[\bar{E}_t^H r_{t+1}^A er_{t+1}\right](2) - \left[\bar{E}_t^F r_{t+1}^A er_{t+1}\right](2)}{\left[E_t(er_{t+1})^2\right](2)} \end{aligned} \tag{45}$$

2.6.3 Third order terms

Take the third order terms of (39)-(39), using (41):

$$\begin{aligned} 0 &= \left[E_t^{Hj} er_{t+1}\right](3) + \tau_t^{Hj}(3) + \left[E_t^{Hj}\left(r_{t+1}^A - \gamma r_{t+1}^{p,Hj}\right) er_{t+1}\right](3) \\ &\quad +\tau\left[E_t^{Hj}\left(r_{t+1}^A - \gamma r_{t+1}^{p,Hj}\right)\right](1) + \frac{1}{2}\left[E_t^{Hj}\left(r_{t+1}^A - \gamma r_{t+1}^{p,Hj}\right)^2 er_{t+1}\right](3) \\ &\quad +\frac{1}{24}\left[E_t^{Hj}\left[er_{t+1}\right]^3\right](3) \end{aligned}$$

and:

$$\begin{aligned} 0 &= \left[E_t^{Fj} er_{t+1}\right](3) - \tau_t^{Fj}(3) + \left[E_t^{Fj}\left(r_{t+1}^A - \gamma r_{t+1}^{p,Fj}\right) er_{t+1}\right](3) \\ &\quad -\tau\left[E_t^{Fj}\left(r_{t+1}^A - \gamma r_{t+1}^{p,Fj}\right)\right](1) + \frac{1}{2}\left[E_t^{Fj}\left(r_{t+1}^A - \gamma r_{t+1}^{p,Fj}\right)^2 er_{t+1}\right](3) \\ &\quad +\frac{1}{24}\left[E_t^{Fj}\left[er_{t+1}\right]^3\right](3) \end{aligned}$$

Next we use the portfolio returns (31)-(32). The first-order portfolio share for a Home agent is:

$$\begin{aligned} & \gamma z_{Hj,t}(1)\left[E_t\left(er_{t+1}\right)^2\right](2) \\ = & \left[E_t^{Hj} er_{t+1}\right](3) + \tau_t^{Hj}(3) + (1-\gamma)\left[E_t^{Hj} r_{t+1}^A er_{t+1}\right](3) \\ & +(1-\gamma)\tau\left[E_t^{Hj} r_{t+1}^A\right](1) - \gamma\frac{2z_{Hj}(0)-1}{2}\left[E_t^{Hj}\left(er_{t+1}\right)^2\right](3) \\ & +\left[-\gamma(1+\gamma)\frac{z_{Hj}(0)\left(1-z_{Hj}(0)\right)}{2} + \frac{1}{6} - \frac{1-\gamma}{2}\frac{1+\gamma}{4}\right]\left[E_t^{Hj}\left[er_{t+1}\right]^3\right](3) \\ & +\frac{(1-\gamma)^2}{2}\left[E_t^{Hj}\left(r_{t+1}^A\right)^2 er_{t+1}\right](3) \\ & -\gamma(1-\gamma)\frac{2z_{Hj}(0)-1}{2}\left[E_t^{Hj} r_{t+1}^A\left(er_{t+1}\right)^2\right](3) \end{aligned} \tag{46}$$

Similarly for a Foreign agent:

$$\begin{aligned}
& \gamma z_{Fj,t}(1)\left[E_t\left(er_{t+1}\right)^2\right](2) \\
= & \left[E_t^{Fj} er_{t+1}\right](3) - \tau_t^{Fj}(3) + (1-\gamma)\left[E_t^{Fj} r_{t+1}^A er_{t+1}\right](3) \\
& -(1-\gamma)\tau\left[E_t^{Fj} r_{t+1}^A\right](1) - \gamma\frac{2z_{Fj}(0)-1}{2}\left[E_t^{Fj}\left(er_{t+1}\right)^2\right](3) \\
& +\left[-\gamma(1+\gamma)\frac{z_{Fj}(0)\left(1-z_{Fj}(0)\right)}{2}+\frac{1}{6}-\frac{1-\gamma}{2}\frac{1+\gamma}{4}\right]\left[E_t^{Fj}\left(er_{t+1}\right)^3\right](3) \\
& +\frac{(1-\gamma)^2}{2}\left[E_t^{Fj}\left(r_{t+1}^A\right)^2 er_{t+1}\right](3) \\
& -\gamma(1-\gamma)\frac{2z_{Fj}(0)-1}{2}\left[E_t^{Fj} r_{t+1}^A\left(er_{t+1}\right)^2\right](3)
\end{aligned} \tag{47}$$

As shown later the third-order expected cubic excess returns are zero for any investors: $\left[E_t^{Hj}\left(er_{t+1}\right)^3\right](3) = \left[E_t^{Fj}\left(er_{t+1}\right)^3\right](3) = 0$. Averaging (46) and (47) across investors in a given country, and combining the results, we get the average portfolio share:

$$\begin{aligned}
& \gamma z_t^A(1)\left[E_t\left(er_{t+1}\right)^2\right](2) \\
= & \frac{1}{2}\left[\left[\bar{E}_t^H er_{t+1}\right](3) + \left[\bar{E}_t^F er_{t+1}\right](3) + \tau_t^D(3)\right] \\
& +\frac{1-\gamma}{2}\left[\left[\bar{E}_t^H r_{t+1}^A er_{t+1}\right](3) + \left[\bar{E}_t^F r_{t+1}^A er_{t+1}\right](3)\right] \\
& +\frac{1-\gamma}{2}\tau\left[\left[\bar{E}_t^H r_{t+1}^A\right](1) - \left[\bar{E}_t^F r_{t+1}^A\right](1)\right] \\
& -\frac{\gamma}{2}\left[\begin{array}{c} \int \frac{2z_{Hj}(0)-1}{2}\left[E_t^{Hj}\left(er_{t+1}\right)^2\right](3)\,dj \\ +\int \frac{2z_{Fj}(0)-1}{2}\left[E_t^{Fj}\left(er_{t+1}\right)^2\right](3)\,dj \end{array}\right] \\
& +\frac{(1-\gamma)^2}{4}\left[\left[\bar{E}_t^H\left(r_{t+1}^A\right)^2 er_{t+1}\right](3) + \left[\bar{E}_t^F\left(r_{t+1}^A\right)^2 er_{t+1}\right](3)\right] \\
& -\frac{\gamma(1-\gamma)}{2}\left[\begin{array}{c} \int \frac{2z_{Hj}(0)-1}{2}\left[E_t^{Hj} r_{t+1}^A\left(er_{t+1}\right)^2\right](3)\,dj \\ +\int \frac{2z_{Fj}(0)-1}{2}\left[E_t^{Fj} r_{t+1}^A\left(er_{t+1}\right)^2\right](3)\,dj \end{array}\right]
\end{aligned} \tag{48}$$

Similarly, taking a cross-country difference leads to:

$$
\begin{aligned}
& \gamma z_t^D (1) \left[E_t \left(er_{t+1} \right)^2 \right] (2) \\
= \; & \left[\bar{E}_t^H er_{t+1} \right] (3) - \left[\bar{E}_t^F er_{t+1} \right] (3) \\
& + (1-\gamma) \left[\left[\bar{E}_t^H r_{t+1}^A er_{t+1} \right] (3) - \left[\bar{E}_t^F r_{t+1}^A er_{t+1} \right] (3) \right] \\
& + (1-\gamma) \tau \left[\left[\bar{E}_t^H r_{t+1}^A \right] (1) + \left[\bar{E}_t^F r_{t+1}^A \right] (1) \right] \\
& -\gamma \left[\begin{array}{c} \int \frac{2z_{Hj}(0)-1}{2} \left[E_t^{Hj} \left(er_{t+1} \right)^2 \right] (3)\, dj \\ -\int \frac{2z_{Fj}(0)-1}{2} \left[E_t^{Fj} \left(er_{t+1} \right)^2 \right] (3)\, dj \end{array} \right] \\
& + \frac{(1-\gamma)^2}{2} \left[\left[\bar{E}_t^H \left(r_{t+1}^A \right)^2 er_{t+1} \right] (3) - \left[\bar{E}_t^F \left(r_{t+1}^A \right)^2 er_{t+1} \right] (3) \right] \\
& -\gamma (1-\gamma) \left[\begin{array}{c} \int \frac{2z_{Hj}(0)-1}{2} \left[E_t^{Hj} r_{t+1}^A \left(er_{t+1} \right)^2 \right] (3)\, dj \\ -\int \frac{2z_{Fj}(0)-1}{2} \left[E_t^{Fj} r_{t+1}^A \left(er_{t+1} \right)^2 \right] (3)\, dj \end{array} \right]
\end{aligned}
\tag{49}
$$

3 Signal extraction

3.1 Conjecture and general formula

The agents infer the future productivity innovations based on the unconditional distribution of shocks, their private signals, and the component of the cross-country difference of asset prices that reflects unobserved future innovations and liquidity shocks:

$$
x_t^D = \varepsilon_{H,t+1} - \varepsilon_{F,t+1} + \lambda \frac{\tau_t^H - \tau_t^F}{\tau} \tag{50}
$$

where λ is a coefficient to be determined. From (10) $\left(\tau_t^H - \tau_t^F\right)/\tau$ is a first-order variable with mean zero and variance $2\theta\sigma_a^2 (1-\rho_\tau)$. It is not correlated with productivity innovations.

We can limit ourselves to a signal extraction based on the cross-country differences. Ultimately, λ is computed by relying on the asset market clearing and the portfolio share (48). The liquidity shocks enter (48) through the various expectations, that involve (50), hence $\tau_t^H - \tau_t^F$, as well as directly as $\tau_t^D (3) = \tau_t^H - \tau_t^F$. The fact that $\tau_t^D (3)$ enters (48) in these two ways is what will allow us to solve for λ. If $\tau_t^A (3)$ was also entering (48) we would need a variable similar to (50) in terms of worldwide average.

A Home investor j wants to infer the vector of productivity innovations $\xi_{t+1} = |\varepsilon_{H,t+1}, \varepsilon_{F,t+1}|'$ using her signals $v_{j,t}^{H,H}$ and $v_{j,t}^{H,F}$, (50), and the unconditional distribution of productivity innovations. The vector of signal is $Y_t = \left| x_t^D, v_{j,t}^{H,H}, v_{j,t}^{H,F}, 0, 0 \right|'$. The vector of signals Y_t is linked to the true innovation vector ξ_{t+1} through a matrix X, and is subjected to a vector of shocks υ: $Y = X\xi + \upsilon$. The errors are independent, hence the variance of υ is a diagonal matrix R.

The generalized least square estimate of ξ_{t+1} denoted by $\hat{\xi}_{t+1}$, follows a Normal distribution with mean ξ_{t+1} and variance $V\left(\hat{\xi}_{t+1}\right)$ where:

$$\hat{\xi}_{t+1} = \left(X'R^{-1}X\right)^{-1} X'R^{-1}Y_t \quad ; \quad V\left(\hat{\xi}_{t+1}\right) = \left(X'R^{-1}X\right)^{-1} \tag{51}$$

3.2 Home investor

For a Home investor j, we use (19) to write the matrices X and R as:

$$X = \begin{vmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{vmatrix} \quad ; \quad diag\left(R\right) = \begin{vmatrix} 4\lambda^2\theta\sigma_a^2 \\ \sigma_{H,H}^2 \\ \sigma_{H,F}^2 \\ \sigma_a^2 \\ \sigma_a^2 \end{vmatrix}$$

We then write:

$$\begin{aligned}
X'R^{-1}Y_t &= \begin{vmatrix} \frac{1}{4\lambda^2\theta\sigma_a^2}x_t^D + \frac{1}{\sigma_{H,H}^2}v_{j,t}^{H,H} \\ -\frac{1}{4\lambda^2\theta\sigma_a^2}x_t^D + \frac{1}{\sigma_{H,F}^2}v_{j,t}^{H,F} \end{vmatrix} \\
X'R^{-1}X &= \begin{vmatrix} \frac{1}{4\lambda^2\theta\sigma_a^2} + \frac{1}{\sigma_{H,H}^2} + \frac{1}{\sigma_a^2} & -\frac{1}{4\lambda^2\theta\sigma_a^2} \\ -\frac{1}{4\lambda^2\theta\sigma_a^2} & \frac{1}{4\lambda^2\theta\sigma_a^2} + \frac{1}{\sigma_{H,F}^2} + \frac{1}{\sigma_a^2} \end{vmatrix} \\
\left(X'R^{-1}X\right)^{-1} &= \frac{\sigma_a^2}{V}\begin{vmatrix} \frac{1}{2\lambda^2\theta} + \left(\frac{\sigma_a^2}{\sigma_{H,F}^2} + 1\right)2 & \frac{1}{2\lambda^2\theta} \\ \frac{1}{2\lambda^2\theta} & \frac{1}{2\lambda^2\theta} + \left(\frac{\sigma_a^2}{\sigma_{H,H}^2} + 1\right)2 \end{vmatrix} \\
V &= \left(\frac{\sigma_a^2}{\sigma_{H,H}^2} + 1\right)\left(\frac{\sigma_a^2}{\sigma_{H,F}^2} + 1\right)2 \\
&\quad + \frac{1}{2\lambda^2\theta}\left(\frac{\sigma_a^2}{\sigma_{H,F}^2} + \frac{\sigma_a^2}{\sigma_{H,H}^2} + 2\right)
\end{aligned} \tag{52}$$

The estimates of the innovations are then:

$$E_t^{Hj}\left(\varepsilon_{H,t+1}\right) = \alpha_{\varepsilon H,xD}^{Hj}x_t^D + \alpha_{\varepsilon H,vH}^{Hj}v_{j,t}^{H,H} + \alpha_{\varepsilon H,vF}^{Hj}v_{j,t}^{H,F} \tag{53}$$

$$E_t^{Hj}\left(\varepsilon_{F,t+1}\right) = \alpha_{\varepsilon F,xD}^{Hj}x_t^D + \alpha_{\varepsilon F,vH}^{Hj}v_{j,t}^{H,H} + \alpha_{\varepsilon F,vF}^{Hj}v_{j,t}^{H,F} \tag{54}$$

where:

$$
\begin{aligned}
\alpha^{Hj}_{\varepsilon H,xD} &= \frac{1}{V}\left(\frac{\sigma_a^2}{\sigma_{H,F}^2}+1\right)\frac{1}{2\lambda^2\theta} \\
\alpha^{Hj}_{\varepsilon H,vH} &= \frac{1}{V}\left[\frac{1}{2\lambda^2\theta}+\left(\frac{\sigma_a^2}{\sigma_{H,F}^2}+1\right)2\right]\frac{\sigma_a^2}{\sigma_{H,H}^2} \\
\alpha^{Hj}_{\varepsilon H,vF} &= \frac{1}{V}\frac{1}{2\lambda^2\theta}\frac{\sigma_a^2}{\sigma_{H,F}^2} \\
\alpha^{Hj}_{\varepsilon F,xD} &= -\frac{1}{V}\left(\frac{\sigma_a^2}{\sigma_{H,H}^2}+1\right)\frac{1}{2\lambda^2\theta} \\
\alpha^{Hj}_{\varepsilon F,vH} &= \frac{1}{V}\frac{1}{2\lambda^2\theta}\frac{\sigma_a^2}{\sigma_{H,H}^2} \\
\alpha^{Hj}_{\varepsilon F,vF} &= \frac{1}{V}\left[\frac{1}{2\lambda^2\theta}+\left(\frac{\sigma_a^2}{\sigma_{H,H}^2}+1\right)2\right]\frac{\sigma_a^2}{\sigma_{H,F}^2}
\end{aligned}
$$

x_t^D only entails first order terms, while $v_{j,t}^{H,H}$ and $v_{j,t}^{H,F}$ entail zero order terms (the idiosyncratic components) and first order terms (the true innovations). We can split the coefficients in (53)-(54) between their various order. $\alpha^{Hj}_{\varepsilon H,xD}$ and $\alpha^{Hj}_{\varepsilon F,xD}$ have zero- and second-order components:

$$
\begin{aligned}
\left[\alpha^{Hj}_{\varepsilon H,xD}\right](0) &= -\left[\alpha^{Hj}_{\varepsilon F,xD}\right](0) = \frac{1}{2\left(1+2\lambda^2\theta\right)} \\
\left[\alpha^{Hj}_{\varepsilon H,xD}\right](1) &= \left[\alpha^{Hj}_{\varepsilon F,xD}\right](1) = 0 \\
\left[\alpha^{Hj}_{\varepsilon H,xD}\right](2) &= -\sigma_a^2\frac{\sigma_{H,F}^2-\sigma_{H,H}^2+4\lambda^2\theta\sigma_{H,F}^2}{4\left(1+2\lambda^2\theta\right)^2\sigma_{H,H}^2\sigma_{H,F}^2} \\
\left[\alpha^{Hj}_{\varepsilon F,xD}\right](2) &= \sigma_a^2\frac{-\sigma_{H,F}^2+\sigma_{H,H}^2+4\lambda^2\theta\sigma_{H,H}^2}{4\left(1+2\lambda^2\theta\right)^2\sigma_{H,H}^2\sigma_{H,F}^2}
\end{aligned}
$$

By contrast, $\alpha^{Hj}_{\varepsilon H,vH}$, $\alpha^{Hj}_{\varepsilon H,vF}$, $\alpha^{Hj}_{\varepsilon F,vH}$ and $\alpha^{Hj}_{\varepsilon F,vF}$ only have a second-order component:

$$
\begin{aligned}
\left[\alpha^{Hj}_{\varepsilon H,vH}\right](2) &= \frac{1+4\lambda^2\theta}{2\left(1+2\lambda^2\theta\right)}\frac{\sigma_a^2}{\sigma_{H,H}^2} \\
\left[\alpha^{Hj}_{\varepsilon H,vF}\right](2) &= \frac{1}{2\left(1+2\lambda^2\theta\right)}\frac{\sigma_a^2}{\sigma_{H,F}^2} \\
\left[\alpha^{Hj}_{\varepsilon F,vH}\right](2) &= \frac{1}{2\left(1+2\lambda^2\theta\right)}\frac{\sigma_a^2}{\sigma_{H,H}^2} \\
\left[\alpha^{Hj}_{\varepsilon F,vF}\right](2) &= \frac{1+4\lambda^2\theta}{2\left(1+2\lambda^2\theta\right)}\frac{\sigma_a^2}{\sigma_{H,F}^2}
\end{aligned}
$$

The various order components of expected productivity innovations (53)-(54) are then:

$$
\begin{aligned}
\left[E_t^{Hj}\left(\varepsilon_{H,t+1}\right)\right](1) &= \frac{1}{2\left(1+2\lambda^2\theta\right)}x_t^D \\
\left[E_t^{Hj}\left(\varepsilon_{F,t+1}\right)\right](1) &= -\frac{1}{2\left(1+2\lambda^2\theta\right)}x_t^D \\
\left[E_t^{Hj}\left(\varepsilon_{H,t+1}\right)\right](2) &= \frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)}\left[\frac{1+4\lambda^2\theta}{\sigma_{H,H}^2}\epsilon_{j,t}^{H,H}+\frac{1}{\sigma_{H,F}^2}\epsilon_{j,t}^{H,F}\right] \\
\left[E_t^{Hj}\left(\varepsilon_{F,t+1}\right)\right](2) &= \frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)}\left[\frac{1}{\sigma_{H,H}^2}\epsilon_{j,t}^{H,H}+\frac{1+4\lambda^2\theta}{\sigma_{H,F}^2}\epsilon_{j,t}^{H,F}\right] \\
\left[E_t^{Hj}\left(\varepsilon_{H,t+1}\right)\right](3) &= -\sigma_a^2\frac{\sigma_{H,F}^2-\sigma_{H,H}^2+4\lambda^2\theta\sigma_{H,F}^2}{4\left(1+2\lambda^2\theta\right)^2\sigma_{H,H}^2\sigma_{H,F}^2}x_t^D \\
&\quad+\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)}\left[\frac{1+4\lambda^2\theta}{\sigma_{H,H}^2}\varepsilon_{H,t+1}+\frac{1}{\sigma_{H,F}^2}\varepsilon_{F,t+1}\right] \\
\left[E_t^{Hj}\left(\varepsilon_{F,t+1}\right)\right](3) &= \sigma_a^2\frac{-\sigma_{H,F}^2+\sigma_{H,H}^2+4\lambda^2\theta\sigma_{H,H}^2}{4\left(1+2\lambda^2\theta\right)^2\sigma_{H,H}^2\sigma_{H,F}^2}x_t^D \\
&\quad+\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)}\left[\frac{1}{\sigma_{H,H}^2}\varepsilon_{H,t+1}+\frac{1+4\lambda^2\theta}{\sigma_{H,F}^2}\varepsilon_{F,t+1}\right]
\end{aligned}
$$

The agent's estimates of the variance-covariance matrix of productivity innovations is equal to $V\left(\hat{\xi}_{t+1}\right)$ in (51), the second-order component of which is:

$$
\left[V\left(\hat{\xi}_{t+1}\right)\right](2) = \frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)}\begin{vmatrix} 1+4\lambda^2\theta & 1 \\ 1 & 1+4\lambda^2\theta \end{vmatrix}
$$

We can also show that the third order component of the variance is zero: $\left[V\left(\hat{\xi}_{t+1}\right)\right](3) = 0$.

We can also compute expectation of higher order products of the innovations. First, recall that:

$$\begin{aligned} E\left(x^2\right) &= (Ex)^2 + var\left(x\right) \\ E\left(xy\right) &= cov\left(x,y\right) + \left(Ex\right)\left(Ey\right) \end{aligned}$$

Using this, we write:

$$\begin{aligned} & \left[E_t^{Hj}\left(\varepsilon_{H,t+1}\right)^2\right](2) \\ = & \left(\left[E_t^{Hj}\left(\varepsilon_{H,t+1}\right)\right](1)\right)^2 + \left[V\left(\hat{\xi}_{t+1}\right)_{1,1}\right](2) \\ = & \left(\frac{1}{2\left(1+2\lambda^2\theta\right)}x_t^D\right)^2 + \frac{1+4\lambda^2\theta}{2\left(1+2\lambda^2\theta\right)}\sigma_a^2 \end{aligned}$$

Similarly:

$$\begin{aligned} \left[E_t^{Hj}\left(\varepsilon_{F,t+1}\right)^2\right](2) &= \left[E_t^{Hj}\left(\varepsilon_{H,t+1}\right)^2\right](2) \\ \left[E_t^{Hj}\left(\varepsilon_{H,t+1}\varepsilon_{F,t+1}\right)\right](2) &= -\left(\frac{1}{2\left(1+2\lambda^2\theta\right)}x_t^D\right)^2 + \left[V\left(\hat{\xi}_{t+1}\right)_{1,2}\right](2) \\ &= -\left(\frac{1}{2\left(1+2\lambda^2\theta\right)}x_t^D\right)^2 + \frac{1}{2\left(1+2\lambda^2\theta\right)}\sigma_a^2 \end{aligned}$$

which implies:

$$\begin{aligned} \left[E_t^{Hj}\left(\varepsilon_{t+1}^D\right)^2\right](2) &= \left[E_t^{Hj}\left(\varepsilon_{H,t+1}\right)^2\right](2) + \left[E_t^{Hj}\left(\varepsilon_{F,t+1}\right)^2\right](2) - 2\left[E_t^{Hj}\left(\varepsilon_{H,t+1}\varepsilon_{F,t+1}\right)\right](2) \\ &= \left(\frac{1}{1+2\lambda^2\theta}x_t^D\right)^2 + \frac{4\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2 \\ \left[E_t^{Hj}\left(\varepsilon_{t+1}^A\right)^2\right](2) &= \frac{1}{4}\left[E_t^{Hj}\left(\varepsilon_{H,t+1}\right)^2\right](2) + \frac{1}{4}\left[E_t^{Hj}\left(\varepsilon_{F,t+1}\right)^2\right](2) + \frac{1}{2}\left[E_t^{Hj}\left(\varepsilon_{H,t+1}\varepsilon_{F,t+1}\right)\right](2) \\ &= \frac{1}{2}\sigma_a^2 \\ \left[E_t^{Hj}\varepsilon_{t+1}^A\varepsilon_{t+1}^D\right](2) &= \frac{1}{2}\left[\left[E_t^{Hj}\left(\varepsilon_{H,t+1}\right)^2\right](2) - \left[E_t^{Hj}\left(\varepsilon_{F,t+1}\right)^2\right](2)\right] = 0 \end{aligned}$$

It is also useful to write third-order expectations of cubic products of the innovations. We use the general property that if x and y are normally distributed, we can write:

$$\begin{aligned} E\left(x^3\right) &= (Ex)^3 + 3\,(Ex)\,var\,(x) \\ y &= x\frac{cov\,(x,y)}{var\,(x)} + \varepsilon_y \end{aligned}$$

where ε_y is independent from x and $E\varepsilon_y = Ey - Ex \cdot cov\,(x,y)\,/var\,(x)$. This implies:

$$\begin{aligned} E\left(yx^2\right) &= E\left(x^3\right)\frac{cov\,(x,y)}{var\,(x)} + E\varepsilon_y E\left(x^2\right) \\ &= E\left(x^3\right)\frac{cov\,(x,y)}{var\,(x)} - \frac{cov\,(x,y)}{var\,(x)}E\left(x^2\right)Ex + EyE\left(x^2\right) \\ &= E\left(x^3\right)\frac{cov\,(x,y)}{var\,(x)} - \frac{cov\,(x,y)}{var\,(x)}\left[var\,(x) + (Ex)^2\right]Ex + EyE\left(x^2\right) \\ &= 2cov\,(x,y)\,Ex + EyE\left(x^2\right) \end{aligned}$$

We therefore write:

$$\begin{aligned} \left[E_t^{Hj}\left(\varepsilon_{H,t+1}\right)^3\right](3) &= \left(\frac{1}{2\left(1+2\lambda^2\theta\right)}x_t^D\right)^3 + 3\left(\frac{1}{2\left(1+2\lambda^2\theta\right)}x_t^D\right)\frac{1+4\lambda^2\theta}{2\left(1+2\lambda^2\theta\right)}\sigma_a^2 \\ \left[E_t^{Hj}\left(\varepsilon_{H,t+1}\right)^2\varepsilon_{F,t+1}\right](3) &= -\left(\frac{1}{2\left(1+2\lambda^2\theta\right)}x_t^D\right)^3 + \frac{1}{2\left(1+2\lambda^2\theta\right)}x_t^D\frac{1-4\lambda^2\theta}{2\left(1+2\lambda^2\theta\right)}\sigma_a^2 \\ \left[E_t^{Hj}\varepsilon_{H,t+1}\left(\varepsilon_{F,t+1}\right)^2\right](3) &= -\left[E_t^{Hj}\left(\varepsilon_{H,t+1}\right)^2\varepsilon_{F,t+1}\right](3) \\ \left[E_t^{Hj}\left(\varepsilon_{F,t+1}\right)^3\right](3) &= -\left[E_t^{Hj}\left(\varepsilon_{H,t+1}\right)^3\right](3) \end{aligned}$$

3.3 Foreign investor

For a Foreign investor j, the vector of signal is $Y_t = \left|x_t^D, v_{j,t}^{F,H}, v_{j,t}^{F,F}, 0, 0\right|'$. The matrix X is identical to the one for a Home investor. We use (20) to write the matrix R as:

$$X = \begin{vmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{vmatrix} \quad ; \quad diag\,(R) = \begin{vmatrix} 4\lambda^2\theta\sigma_a^2 \\ \sigma_{H,F}^2 \\ \sigma_{H,H}^2 \\ \sigma_a^2 \\ \sigma_a^2 \end{vmatrix}$$

We then write:

$$\begin{aligned}
X'R^{-1}Y_t &= \begin{vmatrix} \frac{1}{4\lambda^2\theta\sigma_a^2}x_t^D + \frac{1}{\sigma_{H,F}^2}\upsilon_{j,t}^{F,H} \\ -\frac{1}{4\lambda^2\theta\sigma_a^2}x_t^D + \frac{1}{\sigma_{H,H}^2}\upsilon_{j,t}^{F,F} \end{vmatrix} \\
X'R^{-1}X &= \begin{vmatrix} \frac{1}{4\lambda^2\theta\sigma_a^2} + \frac{1}{\sigma_{H,F}^2} + \frac{1}{\sigma_a^2} & -\frac{1}{4\lambda^2\theta\sigma_a^2} \\ -\frac{1}{4\lambda^2\theta\sigma_a^2} & \frac{1}{4\lambda^2\theta\sigma_a^2} + \frac{1}{\sigma_{H,H}^2} + \frac{1}{\sigma_a^2} \end{vmatrix} \\
\left(X'R^{-1}X\right)^{-1} &= \frac{\sigma_a^2}{V}\begin{vmatrix} \frac{1}{2\lambda^2\theta} + \left(\frac{\sigma_a^2}{\sigma_{H,H}^2} + 1\right)2 & \frac{1}{2\lambda^2\theta} \\ \frac{1}{2\lambda^2\theta} & \frac{1}{2\lambda^2\theta} + \left(\frac{\sigma_a^2}{\sigma_{H,F}^2} + 1\right)2 \end{vmatrix}
\end{aligned}$$

where V is identical to (52). The estimates of the innovations are then:

$$E_t^{Fj}\left(\varepsilon_{H,t+1}\right) = \alpha_{\varepsilon H,xD}^{Fj}x_t^D + \alpha_{\varepsilon H,vH}^{Fj}\upsilon_{j,t}^{F,H} + \alpha_{\varepsilon H,vF}^{Fj}\upsilon_{j,t}^{F,F} \tag{55}$$

$$E_t^{Fj}\left(\varepsilon_{F,t+1}\right) = \alpha_{\varepsilon F,xD}^{Fj}x_t^D + \alpha_{\varepsilon F,vH}^{Fj}\upsilon_{j,t}^{F,H} + \alpha_{\varepsilon F,vF}^{Fj}\upsilon_{j,t}^{F,F} \tag{56}$$

where:

$$\begin{aligned}
\alpha_{\varepsilon H,xD}^{Fj} &= \frac{1}{V}\left(\frac{\sigma_a^2}{\sigma_{H,H}^2} + 1\right)\frac{1}{2\lambda^2\theta} \\
\alpha_{\varepsilon H,vH}^{Fj} &= \frac{1}{V}\left[\frac{1}{2\lambda^2\theta} + \left(\frac{\sigma_a^2}{\sigma_{H,H}^2} + 1\right)2\right]\frac{\sigma_a^2}{\sigma_{H,F}^2} \\
\alpha_{\varepsilon H,vF}^{Fj} &= \frac{1}{V}\frac{1}{2\lambda^2\theta}\frac{\sigma_a^2}{\sigma_{H,H}^2} \\
\alpha_{\varepsilon F,xD}^{Fj} &= -\frac{1}{V}\left(\frac{\sigma_a^2}{\sigma_{H,F}^2} + 1\right)\frac{1}{2\lambda^2\theta} \\
\alpha_{\varepsilon F,vH}^{Fj} &= \frac{1}{V}\frac{1}{2\lambda^2\theta}\frac{\sigma_a^2}{\sigma_{H,F}^2} \\
\alpha_{\varepsilon F,vF}^{Fj} &= \frac{1}{V}\left[\frac{1}{2\lambda^2\theta} + \left(\frac{\sigma_a^2}{\sigma_{H,F}^2} + 1\right)2\right]\frac{\sigma_a^2}{\sigma_{H,H}^2}
\end{aligned}$$

We split the coefficients in (55)-(56) between their various order. $\alpha^{Fj}_{\varepsilon H,xD}$ and $\alpha^{Fj}_{\varepsilon F,xD}$ have zero- and second-order components:

$$\begin{aligned}
\left[\alpha^{Fj}_{\varepsilon H,xD}\right](0) &= -\left[\alpha^{Fj}_{\varepsilon F,xD}\right](0) = \frac{1}{2\left(1+2\lambda^2\theta\right)} \\
\left[\alpha^{Fj}_{\varepsilon H,xD}\right](1) &= \left[\alpha^{Fj}_{\varepsilon F,xD}\right](1) = 0 \\
\left[\alpha^{Fj}_{\varepsilon H,xD}\right](2) &= -\sigma_a^2\frac{-\sigma^2_{H,F}+\sigma^2_{H,H}+4\lambda^2\theta\sigma^2_{H,H}}{4\left(1+2\lambda^2\theta\right)^2\sigma^2_{H,H}\sigma^2_{H,F}} \\
\left[\alpha^{Fj}_{\varepsilon F,xD}\right](2) &= \sigma_a^2\frac{\sigma^2_{H,F}-\sigma^2_{H,H}+4\lambda^2\theta\sigma^2_{H,F}}{4\left(1+2\lambda^2\theta\right)^2\sigma^2_{H,H}\sigma^2_{H,F}}
\end{aligned}$$

By contrast, $\alpha^{Fj}_{\varepsilon H,vH}$, $\alpha^{Fj}_{\varepsilon H,vF}$, $\alpha^{Fj}_{\varepsilon F,vH}$ and $\alpha^{Fj}_{\varepsilon F,vF}$ only have a second-order component:

$$\begin{aligned}
\left[\alpha^{Fj}_{\varepsilon H,vH}\right](2) &= \frac{1+4\lambda^2\theta}{2\left(1+2\lambda^2\theta\right)}\frac{\sigma_a^2}{\sigma^2_{H,F}} \\
\left[\alpha^{Fj}_{\varepsilon H,vF}\right](2) &= \frac{1}{2\left(1+2\lambda^2\theta\right)}\frac{\sigma_a^2}{\sigma^2_{H,H}} \\
\left[\alpha^{Fj}_{\varepsilon F,vH}\right](2) &= \frac{1}{2\left(1+2\lambda^2\theta\right)}\frac{\sigma_a^2}{\sigma^2_{H,F}} \\
\left[\alpha^{Fj}_{\varepsilon F,vF}\right](2) &= \frac{1+4\lambda^2\theta}{2\left(1+2\lambda^2\theta\right)}\frac{\sigma_a^2}{\sigma^2_{H,H}}
\end{aligned}$$

The various order components of expected productivity innovations (55)-(56)

are then:

$$\begin{aligned}
\left[E_t^{Fj}\left(\varepsilon_{H,t+1}\right)\right](1) &= \frac{1}{2\left(1+2\lambda^2\theta\right)}x_t^D \\
\left[E_t^{Fj}\left(\varepsilon_{F,t+1}\right)\right](1) &= -\frac{1}{2\left(1+2\lambda^2\theta\right)}x_t^D \\
\left[E_t^{Fj}\left(\varepsilon_{H,t+1}\right)\right](2) &= \frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)}\left[\frac{1+4\lambda^2\theta}{\sigma_{H,F}^2}\epsilon_{j,t}^{F,H}+\frac{1}{\sigma_{H,H}^2}\epsilon_{j,t}^{F,F}\right] \\
\left[E_t^{Fj}\left(\varepsilon_{F,t+1}\right)\right](2) &= \frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)}\left[\frac{1}{\sigma_{H,F}^2}\epsilon_{j,t}^{F,H}+\frac{1+4\lambda^2\theta}{\sigma_{H,H}^2}\epsilon_{j,t}^{F,F}\right] \\
\left[E_t^{Fj}\left(\varepsilon_{H,t+1}\right)\right](3) &= -\sigma_a^2\frac{-\sigma_{H,F}^2+\sigma_{H,H}^2+4\lambda^2\theta\sigma_{H,H}^2}{4\left(1+2\lambda^2\theta\right)^2\sigma_{H,H}^2\sigma_{H,F}^2}x_t^D \\
&\quad+\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)}\left[\frac{1+4\lambda^2\theta}{\sigma_{H,F}^2}\varepsilon_{H,t+1}+\frac{1}{\sigma_{H,H}^2}\varepsilon_{F,t+1}\right] \\
\left[E_t^{Fj}\left(\varepsilon_{F,t+1}\right)\right](3) &= \sigma_a^2\frac{\sigma_{H,F}^2-\sigma_{H,H}^2+4\lambda^2\theta\sigma_{H,F}^2}{4\left(1+2\lambda^2\theta\right)^2\sigma_{H,H}^2\sigma_{H,F}^2}x_t^D \\
&\quad+\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)}\left[\frac{1}{\sigma_{H,F}^2}\varepsilon_{H,t+1}+\frac{1+4\lambda^2\theta}{\sigma_{H,H}^2}\varepsilon_{F,t+1}\right]
\end{aligned}$$

The agent's estimates of the variance-covariance matrix of productivity innovations is equal to $V\left(\hat{\xi}_{t+1}\right)$ in (51), the second-order component being the same as for the Home investor.

We can also compute the expected values of $\left[E_t^{Fj}\left(\varepsilon_{H,t+1}\right)^2\right](2)$, $\left[E_t^{Fj}\left(\varepsilon_{F,t+1}\right)^2\right](2)$, $\left[E_t^{Fj}\left(\varepsilon_{H,t+1}\varepsilon_{F,t+1}\right)\right](2)$, $\left[E_t^{Fj}\left(\varepsilon_{t+1}^D\right)^2\right](2)$, $\left[E_t^{Fj}\left(\varepsilon_{t+1}^A\right)^2\right](2)$ and $\left[E_t^{Fj}\varepsilon_{t+1}^A\varepsilon_{t+1}^D\right](2)$ which are identical to our results for the Home investor. Similarly, $\left[E_t^{Fj}\left(\varepsilon_{H,t+1}\right)^3\right](3)$, $\left[E_t^{Fj}\left(\varepsilon_{H,t+1}\right)^2\varepsilon_{F,t+1}\right](3)$, $\left[E_t^{Fj}\varepsilon_{H,t+1}\left(\varepsilon_{F,t+1}\right)^2\right](3)$, and $\left[E_t^{Fj}\left(\varepsilon_{F,t+1}\right)^3\right](3)$ are the same as for the Home investor.

3.4 Aggregate expectations

Home and Foreign investors agree to a first-order:

$$\left[E_t^{Hj}\left(\varepsilon_{H,t+1}\right)\right](1)=\left[E_t^{Fj}\left(\varepsilon_{H,t+1}\right)\right](1)=-\left[E_t^{Hj}\left(\varepsilon_{F,t+1}\right)\right](1)=-\left[E_t^{Fj}\left(\varepsilon_{F,t+1}\right)\right](1)$$

When averaging across agents, private signals add up to zero. The aggregate second-order expectations across agents in a country are then all zero.

$$\left[E_t^{Hj}\left(\varepsilon_{H,t+1}\right)\right](2)=\left[E_t^{Fj}\left(\varepsilon_{H,t+1}\right)\right](2)=\left[E_t^{Hj}\left(\varepsilon_{F,t+1}\right)\right](2)=\left[E_t^{Fj}\left(\varepsilon_{F,t+1}\right)\right](2)=0$$

The third-order expectations are the same across agents in a given country. The Home investors' expectations for average productivity and the cross-country differential are:

$$\begin{aligned}
\left[E_t^{Hj}\left(\varepsilon_{H,t+1}+\varepsilon_{F,t+1}\right)\right](3) &= -\sigma_a^2\frac{\sigma_{H,F}^2-\sigma_{H,H}^2}{2\left(1+2\lambda^2\theta\right)\sigma_{H,H}^2\sigma_{H,F}^2}x_t^D \\
&\quad +\sigma_a^2\left[\frac{\varepsilon_{H,t+1}}{\sigma_{H,H}^2}+\frac{\varepsilon_{F,t+1}}{\sigma_{H,F}^2}\right] \\
\left[E_t^{Hj}\left(\varepsilon_{H,t+1}-\varepsilon_{F,t+1}\right)\right](3) &= -\sigma_a^2\frac{\lambda^2\theta\left(\sigma_{H,H}^2+\sigma_{H,F}^2\right)}{\left(1+2\lambda^2\theta\right)^2\sigma_{H,H}^2\sigma_{H,F}^2}x_t^D \\
&\quad +\frac{2\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2\left[\frac{\varepsilon_{H,t+1}}{\sigma_{H,H}^2}-\frac{\varepsilon_{F,t+1}}{\sigma_{H,F}^2}\right]
\end{aligned}$$

Similarly for the Foreign investors:

$$\begin{aligned}
\left[E_t^{Fj}\left(\varepsilon_{H,t+1}+\varepsilon_{F,t+1}\right)\right](3) &= \sigma_a^2\frac{\sigma_{H,F}^2-\sigma_{H,H}^2}{2\left(1+2\lambda^2\theta\right)\sigma_{H,H}^2\sigma_{H,F}^2}x_t^D \\
&\quad +\sigma_a^2\left[\frac{\varepsilon_{H,t+1}}{\sigma_{H,F}^2}+\frac{\varepsilon_{F,t+1}}{\sigma_{H,H}^2}\right] \\
\left[E_t^{Fj}\left(\varepsilon_{H,t+1}-\varepsilon_{F,t+1}\right)\right](3) &= -\sigma_a^2\frac{\lambda^2\theta\left(\sigma_{H,H}^2+\sigma_{H,F}^2\right)}{\left(1+2\lambda^2\theta\right)^2\sigma_{H,H}^2\sigma_{H,F}^2}x_t^D \\
&\quad +\frac{2\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2\left[\frac{\varepsilon_{H,t+1}}{\sigma_{H,F}^2}-\frac{\varepsilon_{F,t+1}}{\sigma_{H,H}^2}\right]
\end{aligned}$$

If signals are more precise on domestic innovations ($\sigma_{H,F}^2>\sigma_{H,H}^2$), and increase in x_t^D leads Home investors to expect a lower worldwide productivity than Foreign investors.

Investors also disagree only when signals are more precise on domestic innovations:

$$\begin{aligned}
\left[E_t^{Hj}\left(\varepsilon_{H,t+1}\right)\right](3)-\left[E_t^{Fj}\left(\varepsilon_{H,t+1}\right)\right](3) &= \Upsilon\left[-x_t^D+\left(1+4\lambda^2\theta\right)\varepsilon_{H,t+1}-\varepsilon_{F,t+1}\right] \\
\left[E_t^{Hj}\left(\varepsilon_{F,t+1}\right)\right](3)-\left[E_t^{Fj}\left(\varepsilon_{F,t+1}\right)\right](3) &= \Upsilon\left[-x_t^D+\varepsilon_{H,t+1}-\left(1+4\lambda^2\theta\right)\varepsilon_{F,t+1}\right]
\end{aligned}$$

where:

$$\Upsilon = \frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)} \frac{\sigma_{H,F}^2 - \sigma_{H,H}^2}{\sigma_{H,H}^2 \sigma_{H,F}^2}$$

4 Solution of the model: general aspects

4.1 Analytical form

The model is solved for seven endogenous variables: the asset prices q_t^D and q_t^A, the capital stocks k_{t+1}^A and k_{t+1}^D, the average portfolio share z_t^A, the consumption of young agents, averages across agents in each country, $c_{y,t}^A$ and $c_{y,t}^D$.

The publicly observed state variables are $S_t = \left[a_t^D, a_t^A, k_t^D, k_t^A\right]'$ as well as x_t^D in (50). We conjecture that the seven variables are quadratic functions of the state variables:

$$\begin{aligned}
q_t^D &= \alpha_{qD} S_t + \alpha_{5,qD} x_t^D + S_t' A_{qD} S_t + \beta_{qD} S_t x_t^D + \mu_{qD} \left(x_t^D\right)^2 + \kappa_{qD} \\
q_t^A &= \alpha_{qA} S_t + \alpha_{5,qA} x_t^D + S_t' A_{qA} S_t + \beta_{qA} S_t x_t^D + \mu_{qA} \left(x_t^D\right)^2 + \kappa_{qA} \\
k_{t+1}^A &= \alpha_{kA} S_t + \alpha_{5,kA} x_t^D + S_t' A_{kA} S_t + \beta_{kA} S_t x_t^D + \mu_{kA} \left(x_t^D\right)^2 + \kappa_{kA} \\
k_{t+1}^D &= \alpha_{kD} S_t + \alpha_{5,kD} x_t^D + S_t' A_{kD} S_t + \beta_{kD} S_t x_t^D + \mu_{kD} \left(x_t^D\right)^2 + \kappa_{kD} \\
c_{yt}^A &= \alpha_{cA} S_t + \alpha_{5,cA} x_t^D + S_t' A_{cA} S_t + \beta_{cA} S_t x_t^D + \mu_{cA} \left(x_t^D\right)^2 + \kappa_{cA} \\
c_{yt}^D &= \alpha_{cD} S_t + \alpha_{5,cD} x_t^D + S_t' A_{cD} S_t + \beta_{cD} S_t x_t^D + \mu_{cD} \left(x_t^D\right)^2
\end{aligned}$$

where the α's are 1x4 matrices of coefficients (with zero and first order components), the α_5's are coefficients (with zero and first order components), the A's are 4x4 matrices of zero-order coefficients, the β's are 1x4 matrices of zero-order coefficients, the μ's are zero-order coefficients, and $\acute{\kappa}$'s are a second-order constants.

The model is solved using seven equations. The capital accumulations (33)-(34), the asset market clearing relation (35)-(36), the consumption Euler equation (37)-(38), alongside (29), and the average between the portfolio Euler relations (39)-(40).

Using (2) and our conjecture, the dynamics of the state variables are:

$$S_{t+1} = N_1 S_t + N_2 \varepsilon_{t+1} + N_3 x_t^D + N_4 \left(x_t^D\right)^2 + N_5 S_t x_t^D + \begin{pmatrix} 0 \\ 0 \\ S_t' N_6 S_t \\ S_t' N_7 S_t \end{pmatrix} + \kappa \quad (57)$$

where $\varepsilon_{t+1} = [\varepsilon_{H,t+1}, \varepsilon_{F,t+1}]'$, $N_4 = [0, 0, \mu_{kD}, \mu_{kA}]'$, $N_6 = A_{kD}$, $N_7 = A_{kA}$ and:

$$N_1 = \begin{vmatrix} \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ \alpha_{1,kD} & \alpha_{2,kD} & \alpha_{3,kD} & \alpha_{4,kD} \\ \alpha_{1,kA} & \alpha_{2,kA} & \alpha_{3,kA} & \alpha_{4,kA} \end{vmatrix}$$

$$N_2 = \begin{vmatrix} 1 & -1 \\ 0.5 & 0.5 \\ 0 & 0 \\ 0 & 0 \end{vmatrix} \quad ; \quad N_3 = \begin{vmatrix} 0 \\ 0 \\ \alpha_{5,kD} \\ \alpha_{5,kA} \end{vmatrix} \quad ; \quad N_5 = \begin{vmatrix} 0_{1x4} \\ 0_{1x4} \\ \beta_{kD} \\ \beta_{kA} \end{vmatrix} \quad ; \quad \kappa = \begin{vmatrix} 0 \\ 0 \\ \kappa_{kD} \\ \kappa_{kA} \end{vmatrix}$$

4.2 Some useful expectations

The optimal portfolio shares combine the expected values of various combinations of the average return (29) and the excess return (30).

4.2.1 Combination of unobserved fundamentals

Using (50), we write that:

$$\left[E_t^{Hj} x_{t+1}^D\right](1) = \left[E_t^{Fj} x_{t+1}^D\right](2) = 0 \tag{58}$$

Furthermore:

$$\left[E_t^{Hj}\left(x_{t+1}^D\right)^2\right](2) = \left[E_t^{Hj}\left(\varepsilon_{t+1}^D + \lambda\frac{\tau_t^D}{\tau}\right)^2\right](2) = 2\sigma_a^2\left(1 + 2\lambda^2\theta\right) \tag{59}$$

4.2.2 $\left[E_t^{Hj}\left(er_{t+1}\right)^2\right](2)$

We first compute the second order component of the expected quadratic excess returns, which relies only on the linear terms in (30):

$$\begin{aligned}
& \left[E_t^{Hj}\left(er_{t+1}\right)^2\right](2) \\
= & \left[E_t^{Hj}\left((1-r_q)\left[a_{t+1}^D - \omega k_{t+1}^D\right] + r_q q_{t+1}^D - q_t^D\right)^2\right](2) \\
= & \left[E_t^{Hj}\left(\begin{array}{c} (1-r_q)\left[a_{t+1}^D - \omega k_{t+1}^D\right] \\ +r_q\left[\alpha_{1,qD}(0)\, a_{t+1}^D + \alpha_{3,qD}(0)\, k_{t+1}^D + \alpha_{5,qD}(0)\, x_{t+1}^D\right] - q_t^D \end{array}\right)^2\right](2) \\
= & \left[E_t^{Hj}\left(\begin{array}{c} \left[1 - r_q + r_q\alpha_{1,qD}(0)\right]\varepsilon_{t+1}^D + r_q\alpha_{5,qD}(0)\, x_{t+1}^D \\ +\left[1 - r_q + r_q\alpha_{1,qD}(0)\right]\rho a_t^D + \left[r_q\alpha_{3,qD}(0) - \omega(1-r_q)\right]k_{t+1}^D - q_t^D \end{array}\right)^2\right](2)
\end{aligned}$$

Recall that expected excess returns are zero to a first-order, which implies:

$$
\begin{aligned}
0 &= \left[E_t^{Hj} er_{t+1}\right](1) \\
&= (1-r_q)\left[\left[E_t^{Hj} a_{t+1}^D\right](1) - \omega k_{t+1}^D(1)\right] + r_q\left[E_t^{Hj} q_{t+1}^D\right](1) - q_t^D(1) \\
&= \left[1-r_q+r_q\alpha_{1,qD}(0)\right]\rho a_t^D + \left[r_q\alpha_{3,qD}(0) - \omega(1-r_q)\right]k_{t+1}^D(1) - q_t^D(1) \\
&\quad + \left[1-r_q+r_q\alpha_{1,qD}(0)\right]\left[E_t^{Hj}\varepsilon_{t+1}^D\right](1) + r_q\alpha_{5,qD}(0)\left[E_t^{Hj} x_{t+1}^D\right](1) \\
&= \left[1-r_q+r_q\alpha_{1,qD}(0)\right]\rho a_t^D + \left[r_q\alpha_{3,qD}(0) - \omega(1-r_q)\right]k_{t+1}^D(1) - q_t^D(1) \\
&\quad + \frac{1-r_q+r_q\alpha_{1,qD}(0)}{1+2\lambda^2\theta} x_t^D(1)
\end{aligned}
$$

where we used (58). We can then write:

$$
\begin{aligned}
&\left[E_t^{Hj}\left(er_{t+1}\right)^2\right](2) \\
=\; &\left[E_t^{Hj}\left(\left[1-r_q+r_q\alpha_{1,qD}(0)\right]\varepsilon_{t+1}^D + r_q\alpha_{5,qD}(0)\,x_{t+1}^D\right)^2\right](2) \\
&-2\frac{1-r_q+r_q\alpha_{1,qD}(0)}{1+2\lambda^2\theta} x_t^D(1)\left[E_t^{Hj}\begin{pmatrix}\left[1-r_q+r_q\alpha_{1,qD}(0)\right]\varepsilon_{t+1}^D \\ +r_q\alpha_{5,qD}(0)\,x_{t+1}^D\end{pmatrix}\right](1) \\
&+\left(\frac{1-r_q+r_q\alpha_{1,qD}(0)}{1+2\lambda^2\theta} x_t^D(1)\right)^2 \\
=\; &\left[E_t^{Hj}\left(\left[1-r_q+r_q\alpha_{1,qD}(0)\right]\varepsilon_{t+1}^D + r_q\alpha_{5,qD}(0)\,x_{t+1}^D\right)^2\right](2) \\
&-\left(\frac{1-r_q+r_q\alpha_{1,qD}(0)}{1+2\lambda^2\theta} x_t^D(1)\right)^2
\end{aligned}
$$

Using (59) and our results for $\left[E_t^{Hj}\left(\varepsilon_{t+1}^D\right)^2\right](2)$ we obtain:

$$
\begin{aligned}
&\left[E_t^{Hj}\left(er_{t+1}\right)^2\right](2) \\
=\; &2\left[\left(1-r_q+r_q\alpha_{1,qD}(0)\right)^2\frac{2\lambda^2\theta}{1+2\lambda^2\theta} + \left(r_q\alpha_{5,qD}(0)\right)^2\left(1+2\lambda^2\theta\right)\right]\sigma_a^2
\end{aligned}
$$

The expectation for a foreign investor is identical. Using the form of $\alpha_{5,qD}(0)$, derived below, this can be rewritten as:

$$
\begin{aligned}
&\left[E_t^{Hj}\left(er_{t+1}\right)^2\right](2) \qquad (60) \\
=\; &2\frac{\left(1-r_q+r_q\alpha_{1,qD}(0)\right)^2}{1+2\lambda^2\theta}\left[2\lambda^2\theta + \left(\frac{r_q}{1+\left[(1-r_q)\,\omega - r_q\alpha_{3,qD}(0)\right]\frac{1}{\xi}}\right)^2\right]\sigma_a^2
\end{aligned}
$$

4.2.3 $\left[E_t^{Hj} r_{t+1}^A er_{t+1}\right](2)$

We now compute the second order component of $\left[E_t^{Hj} r_{t+1}^A er_{t+1}\right](2)$, which relies only on the linear terms in (29) and the excess return (30):

$$\begin{aligned}
&\left[E_t^{Hj} r_{t+1}^A er_{t+1}\right](2) \\
= &\left[E_t^{Hj}\left[\begin{array}{c}(1-r_q)\left[a_{t+1}^A-\omega k_{t+1}^A\right]\\ +r_q q_{t+1}^A - q_t^A\end{array}\right]\left[\begin{array}{c}(1-r_q)\left[a_{t+1}^D-\omega k_{t+1}^D\right]\\ +r_q q_{t+1}^D - q_t^D\end{array}\right]\right](2) \\
= &\left(1-r_q+r_q\alpha_{2,qA}(0)\right)\left[E_t^{Hj}\varepsilon_{t+1}^A\left[(1-r_q)\left[a_{t+1}^D-\omega k_{t+1}^D\right]+r_q q_{t+1}^D-q_t^D\right]\right](2) \\
&+\left[\begin{array}{c}\left(1-r_q+r_q\alpha_{2,qA}(0)\right)\rho a_t^A(1)\\ +\left[r_q\alpha_{4,qA}(0)-\omega(1-r_q)\right]k_{t+1}^A(1)\\ -q_t^A(1)\end{array}\right]\left[E_t^{Hj}\left[\begin{array}{c}(1-r_q)\left[a_{t+1}^D-\omega k_{t+1}^D\right]\\ +r_q q_{t+1}^D-q_t^D\end{array}\right]\right](1)
\end{aligned}$$

where we use our first order results derived below. Notice that the last bracket in the second row is equal to the first-order expected excess return, which is zero. We therefore get:

$$\begin{aligned}
&\left[E_t^{Hj} r_{t+1}^A er_{t+1}\right](2) \\
= &\left(1-r_q+r_q\alpha_{2,qA}(0)\right)\left[\begin{array}{c}\left[1-r_q+r_q\alpha_{1,qD}(0)\right]\rho a_t^D(1)\\ +\left[r_q\alpha_{3,qD}(0)-\omega(1-r_q)\right]k_{t+1}^D(1)\\ -q_t^D(1)\end{array}\right]\left[E_t^{Hj}\left[\varepsilon_{t+1}^A\right]\right](1) \\
&+\left(1-r_q+r_q\alpha_{2,qA}(0)\right)\left[E_t^{Hj}\varepsilon_{t+1}^A\left[\begin{array}{c}\left[1-r_q+r_q\alpha_{1,qD}(0)\right]\varepsilon_{t+1}^D\\ +r_q\alpha_{5,qD}(0)\,x_{t+1}^D\end{array}\right]\right](2)
\end{aligned}$$

From the signal extraction results we know that $\left[E_t^{Hj}\left[\varepsilon_{t+1}^A\right]\right](1)=0$. In addition ε_{t+1}^A and x_{t+1}^D are independent. We are therefore left with $\left[E_t^{Hj}\varepsilon_{t+1}^A\varepsilon_{t+1}^D\right](2)$, which is zero as shown in the signal extraction. Therefore:

$$\left[E_t^{Hj} r_{t+1}^A er_{t+1}\right](2)=0 \tag{61}$$

for all investors in all countries.

4.2.4 Second order expected excess returns

Using (61) in (44) we get:

$$
\begin{aligned}
0 &= \left[\bar{E}_t^H er_{t+1}\right](2) + \left[\bar{E}_t^F er_{t+1}\right](2) \\
&\quad + (1-\gamma)\left[\bar{E}_t^H r_{t+1}^A er_{t+1}\right](2) + (1-\gamma)\left[\bar{E}_t^F r_{t+1}^A er_{t+1}\right](2) \\
&= \left[\bar{E}_t^H er_{t+1}\right](2) + \left[\bar{E}_t^F er_{t+1}\right](2) \qquad (62)
\end{aligned}
$$

5 First-order solution

To a first-order, we write the form of the asset price difference, the capital accumulations (33)-(34), the asset market clearing relation (35) as:

$$
\begin{aligned}
q_t^D(1) &= \alpha_{qD}(0)\,S_t(1) + \alpha_{5,qD}(0)\,x_t^D(1) \\
k_{t+1}^A(1) - k_t^A(1) &= \frac{1}{\xi} q_t^A(1) \\
k_{t+1}^D(1) - k_t^D(1) &= \frac{1}{\xi} q_t^D(1) \\
q_t^A(1) + k_{t+1}^A(1) &= \frac{1}{1-\bar{c}} a_t^A(1) + \frac{1-\omega}{1-\bar{c}} k_t^A(1) - \frac{\bar{c}}{1-\bar{c}} c_{yt}^A(1)
\end{aligned}
$$

(39)-(40) imply that expected excess returns are zero:

$$
\left[E_t^{Hj} er_{t+1}\right](1) = \left[E_t^{Fj} er_{t+1}\right](1) = 0
$$

The consumption Euler equation (37)-(38) involve expectations of r_{t+1}^A, which boils down to expectations of q_{t+1}^A and a_{t+1}^A. Under our conjecture for the form of the solution, expectations of q_{t+1}^A consist of expectations of a_{t+1}^A and a_{t+1}^D, the capital stocks at time $t+1$, which are known, and expectations of x_{t+1}^D from (50), which are zero for all agents as there is no information at time t on productivity innovations at time $t+2$ and on liquidity shocks at time $t+1$. The expectations thus boil down to first-order expectations of productivity innovations, which are the same for all agents as shown in the signal extraction analysis. This implies that:

$$
\left[E_t^{Hj} r_{t+1}^A\right](1) = \left[E_t^{Fj} r_{t+1}^A\right](1) = \left[E_t r_{t+1}^A\right](1)
$$

The consumption Euler equation (37)-(38) thus become:

$$
\begin{aligned}
(1-\gamma)\left[E_t r_{t+1}^A\right](1) &= \frac{\gamma}{1-\bar{c}}\left[a_t^A(1) + (1-\omega)k_t^A(1) - c_{yt}^A(1)\right] \\
0 &= a_t^D(1) + (1-\omega)k_t^D(1) - c_{yt}^D(1)
\end{aligned}
$$

From (29) we write:

$$\left[E_t r_{t+1}^A\right](1) = (1 - r_q)\left[\rho a_t^A(1) - \omega k_{t+1}^A(1)\right] + r_q\left[E_t q_{t+1}^A\right](1) - q_t^A(1) \qquad (63)$$

where we used the results from the signal extraction, namely: $\left[E_t \varepsilon_{t+1}^A\right](1) = 0$.

5.1 Worldwide averages

We start with the solution for worldwide averages. From (33) and (35) we write the average asset price and consumption as function of the average capital:

$$\begin{aligned}
q_t^A(1) &= \xi\left[k_{t+1}^A(1) - k_t^A(1)\right] = \xi\left[\alpha_{kA}(0) S_t(1) + \alpha_{5,kA}(0) x_t^D(1) - k_t^A(1)\right] \\
&= \xi\left[\alpha_{1,kA}(0) a_t^D(1) + \alpha_{3,kA}(0) k_t^D(1) + \alpha_{5,kA}(0) x_t^D(1)\right] \\
&\quad +\xi\left[\alpha_{2,kA}(0) a_t^A(1) + (\alpha_{4,kA}(0) - 1) k_t^A(1)\right] \\
c_{yt}^A(1) &= \frac{1}{\bar{c}} a_t^A(1) + \frac{1-\bar{c}}{\bar{c}}\left(\frac{1-\omega}{1-\bar{c}} + \xi\right) k_t^A(1) - \frac{1-\bar{c}}{\bar{c}}(1+\xi) k_{t+1}^A(1)
\end{aligned}$$

Using these results and (63), (37) becomes:

$$\begin{aligned}
0 &= -a_t^A(1) - (1-\omega+\xi) k_t^A(1) + (1+\xi) k_{t+1}^A(1) - \bar{c}\frac{1-\gamma}{\gamma}\left[E_t r_{t+1}^A\right](1) \\
0 &= -a_t^A(1) - (1-\omega+\xi) k_t^A(1) + (1+\xi) k_{t+1}^A(1) \\
&\quad -\bar{c}\frac{1-\gamma}{\gamma}(1-r_q)\left[\rho a_t^A(1) - \omega k_{t+1}^A(1)\right] \\
&\quad -\bar{c}\frac{1-\gamma}{\gamma} r_q \xi\left[\alpha_{2,kA}(0) \rho a_t^A(1) + (\alpha_{4,kA}(0) - 1) k_{t+1}^A(1)\right] \\
&\quad +\bar{c}\frac{1-\gamma}{\gamma}\xi\left[\alpha_{2,kA}(0) a_t^A(1) + (\alpha_{4,kA}(0) - 1) k_t^A(1)\right] \\
&\quad -\bar{c}\frac{1-\gamma}{\gamma} r_q \xi\left[\alpha_{1,kA}(0) \rho a_t^D(1) + \frac{\alpha_{1,kA}(0)}{1+2\lambda^2\theta} x_t^D(1) + \alpha_{3,kA}(0) k_{t+1}^D(1)\right] \\
&\quad +\bar{c}\frac{1-\gamma}{\gamma}\xi\left[\alpha_{1,kA}(0) a_t^D(1) + \alpha_{3,kA}(0) k_t^D(1) + \alpha_{5,kA}(0) x_t^D(1)\right]
\end{aligned}$$

where we used (58), $\left[E_t \varepsilon_{t+1}^A\right](1) = 0$, $\left[E_t \varepsilon_{t+1}^D\right](1) = \left[1 + 2\lambda^2\theta\right]^{-1} x_t^D(1)$. Using our conjecture for $k_{t+1}^A(1)$ this becomes:

$$\begin{aligned}
0 &= \left[\begin{array}{c} -1-\bar{c}\frac{1-\gamma}{\gamma}(1-r_q)\rho+\bar{c}\frac{1-\gamma}{\gamma}\xi\alpha_{2,kA}(0)-\bar{c}\frac{1-\gamma}{\gamma}r_q\xi\alpha_{2,kA}(0)\rho \\ +\left[1+\xi+\bar{c}\frac{1-\gamma}{\gamma}(1-r_q)\omega+\bar{c}\frac{1-\gamma}{\gamma}r_q\xi(1-\alpha_{4,kA}(0))\right]\alpha_{2,kA}(0)\end{array}\right]a_t^A(1) \\
&+\left[\begin{array}{c}\omega\left[1+\bar{c}\frac{1-\gamma}{\gamma}(1-r_q)\right] \\ -\left[1+\xi+\bar{c}\frac{1-\gamma}{\gamma}(1-r_q)(\omega+\xi)\right](1-\alpha_{4,kA}(0)) \\ -\bar{c}\frac{1-\gamma}{\gamma}r_q\xi(1-\alpha_{4,kA}(0))^2\end{array}\right]k_t^A(1) \\
&-\bar{c}\frac{1-\gamma}{\gamma}r_q\xi\left[\alpha_{1,kA}(0)\rho a_t^D(1)+\frac{\alpha_{1,kA}(0)}{1+\lambda^2\theta(1-\rho_\tau)}x_t^D(1)+\alpha_{3,kA}(0)k_{t+1}^D(1)\right] \\
&+\bar{c}\frac{1-\gamma}{\gamma}\xi\left[\alpha_{1,kA}(0)a_t^D(1)+\alpha_{3,kA}(0)k_t^D(1)+\alpha_{5,kA}(0)x_t^D(1)\right] \\
&+\left[\begin{array}{c}1+\xi+\bar{c}\frac{1-\gamma}{\gamma}(1-r_q)\omega \\ +\bar{c}\frac{1-\gamma}{\gamma}r_q\xi(1-\alpha_{4,kA}(0))\end{array}\right]\left[\alpha_{1,kA}(0)a_t^D(1)+\alpha_{3,kA}(0)k_t^D(1)\right]
\end{aligned}$$

The coefficients on $a_t^D(1)$, $x_t^D(1)$, $k_t^D(1)$ and $k_{t+1}^D(1)$ are zero when $\alpha_{1,kA}(0) = \alpha_{3,kA}(0) = 0$. Setting the coefficient on $k_t^A(1)$ to zero gives a second order polynomial in $\alpha_{4,kA}(0)$:

$$\begin{aligned}
0 &= P[\alpha_{4,kA}(0)] \\
&= -\left[1-\omega+\xi\left(1+\bar{c}\frac{1-\gamma}{\gamma}\right)\right]-\bar{c}\frac{1-\gamma}{\gamma}r_q\xi(\alpha_{4,kA}(0))^2 \qquad (64) \\
&\quad+\left[1+\bar{c}\frac{1-\gamma}{\gamma}(1-r_q)\omega+\left(1+\bar{c}\frac{1-\gamma}{\gamma}(1+r_q)\right)\xi\right]\alpha_{4,kA}(0)
\end{aligned}$$

Notice that:

$$\begin{aligned}
P[-\infty] &< 0 \\
P[0] &< 0 \\
P[1] &= \omega\left[1+\bar{c}\frac{1-\gamma}{\gamma}(1-r_q)\right]>0 \\
P[+\infty] &< 0
\end{aligned}$$

The Polynomial therefore has one root above one, and the other between 0 and 1, which is the only one for which the system is stable. Setting the coefficient on $a_t^A(1)$ to zero gives the solution for $\alpha_{2,kA}(0)$. The zero-order coefficients for the worldwide averages of capital are thus:

$$\begin{aligned}
\alpha_{1,kA}(0) &= \alpha_{3,kA}(0) = \alpha_{5,kA}(0) = 0 \\
\alpha_{4,kA}(0) &\in (0,1) \qquad (65) \\
\alpha_{2,kA}(0) &= \frac{\gamma + \bar{c}(1-\gamma)(1-r_q)\rho}{\gamma(1+\xi) + \bar{c}(1-\gamma)\left[(1-r_q)\omega + r_q\xi(1-\alpha_{4,kA}(0)) + \xi(1-r_q\rho)\right]} > 0
\end{aligned}$$

where $\alpha_{4,kA}(0)$ corresponds to the root of (64) that is between zero and one.

The zero-order coefficients for the worldwide averages of asset prices immediately follow:

$$\begin{aligned}
\alpha_{1,qA}(0) &= \alpha_{3,qA}(0) = \alpha_{5,qA}(0) = 0 \\
\alpha_{2,qA}(0) &= \xi\alpha_{2,kA}(0) > 0 \qquad (66) \\
\alpha_{4,qA}(0) &= \xi(\alpha_{4,kA}(0) - 1) < 0
\end{aligned}$$

Similarly the zero-order coefficients for the worldwide averages of consumption are:

$$\begin{aligned}
\alpha_{1,cA}(0) &= \alpha_{3,cA}(0) = \alpha_{5,cA}(0) = 0 \\
\alpha_{2,cA}(0) &= \frac{1}{\bar{c}}\left[1 - (1-\bar{c})(1+\xi)\alpha_{2,kA}(0)\right] \qquad (67) \\
\alpha_{4,cA}(0) &= \frac{1-\bar{c}}{\bar{c}}\left(\frac{1-\omega}{1-\bar{c}} + \xi - (1+\xi)\alpha_{4,kA}(0)\right)
\end{aligned}$$

5.2 Cross-country differences

In terms of cross-country differences, the solution for consumption is given directly by (38):

$$\begin{aligned}
\alpha_{2,cD}(0) &= \alpha_{4,cD}(0) = \alpha_{5,cD}(0) = 0 \\
\alpha_{1,cD}(0) &= 1 \qquad (68) \\
\alpha_{3,cD}(0) &= 1 - \omega
\end{aligned}$$

The coefficients on $k_{t+1}^D(1)$, conditional on the coefficients on $q_t^D(1)$, are given by (34):

$$\begin{aligned}
\alpha_{1,kD}(0) &= \frac{1}{\xi}\alpha_{1,qD}(0) \\
\alpha_{2,kD}(0) &= \frac{1}{\xi}\alpha_{2,qD}(0) \\
\alpha_{3,kD}(0) &= 1+\frac{1}{\xi}\alpha_{3,qD}(0) \\
\alpha_{4,kD}(0) &= \frac{1}{\xi}\alpha_{4,qD}(0) \\
\alpha_{5,kD}(0) &= \frac{1}{\xi}\alpha_{5,qD}(0)
\end{aligned}$$

We solve for the coefficients on $q_t^D(1)$ using the fact that first-order expected excess returns are zero. Using (30) and our conjecture for $q_t^D(1)$ this implies:

$$\begin{aligned}
0 &= (1-r_q)\left[\left[E_t^{Hj}a_{t+1}^D\right](1)-\omega k_{t+1}^D(1)\right]+r_q\left[E_t^{Hj}q_{t+1}^D\right](1)-q_t^D(1) \\
&= \left[(1-r_q)+r_q\alpha_{1,qD}(0)\right]\left[E_t^{Hj}a_{t+1}^D\right](1) \\
&\quad +\left[r_q\alpha_{3,qD}(0)-(1-r_q)\,\omega\right]k_{t+1}^D(1)+r_q\alpha_{2,qD}(0)\left[E_t^{Hj}a_{t+1}^A\right](1) \\
&\quad +r_q\alpha_{4,qD}(0)\,k_{t+1}^A(1)+r_q\alpha_{5,qD}(0)\left[E_t^{Hj}x_{t+1}^D\right](1)-q_t^D(1) \\
&= \left[(1-r_q)+r_q\alpha_{1,qD}(0)\right]\rho a_t^D(1)+\frac{(1-r_q)+r_q\alpha_{1,qD}(0)}{1+2\lambda^2\theta}x_t^D(1) \\
&\quad +\left[r_q\alpha_{3,qD}(0)-(1-r_q)\,\omega\right]k_{t+1}^D(1)+r_q\alpha_{2,qD}(0)\,\rho a_t^A(1) \\
&\quad +r_q\alpha_{4,qD}(0)\,k_{t+1}^A(1)-q_t^D(1)
\end{aligned}$$

where we used $\left[E_t x_{t+1}^D\right](1)=0$, $\left[E_t\varepsilon_{t+1}^A\right](1)=0$, $\left[E_t\varepsilon_{t+1}^D\right](1)=\left[1+2\lambda^2\theta\right]^{-1}x_t^D(1)$. Using our results for the worldwide averages and the coefficients on $k_{t+1}^D(1)$, conditional on the coefficients on $q_t^D(1)$, this becomes:

$$
\begin{aligned}
0 = & \left[\begin{array}{c} \left[(1-r_q)+r_q\alpha_{1,qD}(0)\right]\rho \\ +\left[r_q\alpha_{3,qD}(0)-(1-r_q)\omega-\xi\right]\frac{1}{\xi}\alpha_{1,qD}(0) \end{array} \right] a_t^D(1) \\
& +\left[\begin{array}{c} r_q\alpha_{2,qD}(0)\rho+r_q\alpha_{4,qD}(0)\alpha_{2,kA}(0) \\ +\left[r_q\alpha_{3,qD}(0)-(1-r_q)\omega-\xi\right]\frac{1}{\xi}\alpha_{2,qD}(0) \end{array} \right] a_t^A(1) \\
& +\left[\left[r_q\alpha_{3,qD}(0)-(1-r_q)\omega\right]\left(1+\frac{1}{\xi}\alpha_{3,qD}(0)\right)-\alpha_{3,qD}(0)\right] k_t^D(1) \qquad (69) \\
& +\left[r_q\alpha_{4,qD}(0)\alpha_{4,kA}(0)+\left[r_q\alpha_{3,qD}(0)-(1-r_q)\omega-\xi\right]\frac{1}{\xi}\alpha_{4,qD}(0)\right] k_t^A(1) \\
& +\left[\frac{1-r_q+r_q\alpha_{1,qD}(0)}{1+2\lambda^2\theta}+\left[r_q\alpha_{3,qD}(0)-(1-r_q)\omega-\xi\right]\frac{1}{\xi}\alpha_{5,qD}(0)\right] x_t^D(1)
\end{aligned}
$$

Setting the coefficient on $k_t^D(1)$ in (69) to zero we get a quadratic polynomial in $\alpha_{3,qD}(0)$:

$$0=(1-r_q)\xi\omega+(1-r_q)(\xi+\omega)\alpha_{3,qD}(0)-r_q(\alpha_{3,qD}(0))^2 \qquad (70)$$

To facilitate the analysis, we rewrite (70) as a polynomial in $\alpha_{3,kD}(0)$, using the fact that $\alpha_{3,qD}(0)=\xi\left[\alpha_{3,kD}(0)-1\right]$:

$$
\begin{aligned}
0 &= P\left[\alpha_{3,kD}(0)\right] \\
&= -\xi+\left[(1-r_q)(\xi+\omega)+2r_q\xi\right]\alpha_{3,kD}(0)-r_q\xi(\alpha_{3,kD}(0))^2
\end{aligned}
$$

Notice that:

$$
\begin{aligned}
P[-\infty] &< 0 \\
P[0] &< 0 \\
P[1] &= (1-r_q)\omega>0 \\
P[+\infty] &< 0
\end{aligned}
$$

The Polynomial therefore has one root above one, and the other between 0 and 1, which is the only one for which the system is stable. As $\alpha_{3,kD}(0)\in(0,1)$, $\alpha_{3,qD}(0)=\xi\left[\alpha_{3,kD}(0)-1\right]$ is then the negative root of (70).

$$\alpha_{3,qD}(0)=\frac{1}{2r_q}\left[(1-r_q)(\xi+\omega)-\left[(1-r_q)^2(\xi+\omega)^2+4r_q(1-r_q)\xi\omega\right]^{0.5}\right]$$

Setting the coefficient on $k_t^A(1)$ in (69) to zero we get:

$$0 = \left[-r_q \left(1 - \alpha_{4,kA}(0)\right) - (1 - r_q)\left(1 + \frac{\omega}{\xi}\right) + \frac{r_q}{\xi}\alpha_{3,qD}(0) \right] \alpha_{4,qD}(0)$$

As $\alpha_{4,kA}(0) \in (0,1)$ and $\alpha_{3,qD}(0) < 0$, the bracket is negative, implying that $\alpha_{4,qD}(0) = 0$.

Setting the coefficient on $a_t^A(1)$ in (69) to zero, using our result that $\alpha_{4,qD}(0) = 0$, we get:

$$0 = \left[-(1 - r_q\rho) - (1 - r_q)\frac{\omega}{\xi} + \frac{r_q}{\xi}\alpha_{3,qD}(0) \right] \alpha_{2,qD}(0)$$

As $\alpha_{3,qD}(0) < 0$, the bracket is negative, implying that $\alpha_{2,qD}(0) = 0$.

Setting the coefficient on $a_t^D(1)$ in (69) to zero, we get:

$$\alpha_{1,qD}(0) = \frac{(1 - r_q)\rho}{1 - r_q\rho + \left[(1 - r_q)\omega - r_q\alpha_{3,qD}(0)\right]\frac{1}{\xi}}$$

Setting the coefficient on $x_t^D(1)$ in (69) to zero, we get:

$$\alpha_{5,qD}(0) = \frac{1 - r_q + r_q\alpha_{1,qD}(0)}{1 + \left[(1 - r_q)\omega - r_q\alpha_{3,qD}(0)\right]\frac{1}{\xi}} \frac{1}{1 + 2\lambda^2\theta}$$

We have now solved for the zero-order coefficients of the cross-country differences. The coefficients on $q_t^D(1)$ are:

$$\begin{aligned}
\alpha_{2,qD}(0) &= \alpha_{4,qD}(0) = 0 \\
\alpha_{3,qD}(0) &= \frac{1}{2r_q}\left[(1 - r_q)(\xi + \omega) - \left[(1 - r_q)^2 (\xi + \omega)^2 + 4r_q(1 - r_q)\xi\omega \right]^{0.5} \right] \\
\alpha_{1,qD}(0) &= \frac{(1 - r_q)\rho}{1 - r_q\rho + \left[(1 - r_q)\omega - r_q\alpha_{3,qD}(0)\right]\frac{1}{\xi}} \\
\alpha_{5,qD}(0) &= \frac{1 - r_q + r_q\alpha_{1,qD}(0)}{1 + \left[(1 - r_q)\omega - r_q\alpha_{3,qD}(0)\right]\frac{1}{\xi}} \frac{1}{1 + \lambda^2\theta(1 - \rho_\tau)}
\end{aligned} \tag{71}$$

The coefficients on $k_{t+1}^D(1)$ are:

$$\begin{aligned}
\alpha_{2,kD}(0) &= \alpha_{4,kD}(0) = 0 \\
\alpha_{1,kD}(0) &= \frac{(1 - r_q)\rho}{\xi(1 - r_q\rho) + (1 - r_q)\omega - r_q\alpha_{3,qD}(0)} \\
\alpha_{3,kD}(0) &= 1 + \frac{1}{\xi}\alpha_{3,qD}(0) \in (0,1) \\
\alpha_{5,kD}(0) &= \frac{1 - r_q + r_q\alpha_{1,qD}(0)}{\xi + (1 - r_q)\omega - r_q\alpha_{3,qD}(0)} \frac{1}{1 + 2\lambda^2\theta}
\end{aligned} \tag{72}$$

Note that these coefficients are functions of λ that we have yet to solve for. The coefficients on $c_{yt}^D(1)$ are given by (68).

5.3 Expectations of state variables

The dynamics of the first-order state variables are given by (57):

$$S_{t+1}(1) = N_1(0) S_t(1) + N_2 \varepsilon_{t+1}(1) + N_3(0) x_t^D(1)$$

where:

$$N_1(0) = \begin{vmatrix} \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ \alpha_{1,kD}(0) & 0 & \alpha_{3,kD}(0) & 0 \\ 0 & \alpha_{2,kA}(0) & 0 & \alpha_{4,kA}(0) \end{vmatrix} \quad ; \quad N_3(0) = \begin{vmatrix} 0 \\ 0 \\ \alpha_{5,kD}(0) \\ 0 \end{vmatrix}$$

The first-order expectation of future state variables by a Home investor is:

$$\begin{aligned} \left[E_t^{Hj} S_{t+1}\right](1) &= N_1(0) S_t(1) + \begin{vmatrix} \frac{1}{1+2\lambda^2\theta} \\ 0 \\ \alpha_{5,kD}(0) \\ 0 \end{vmatrix} x_t^D(1) \\ &= N_1(0) S_t(1) + \left[N_2 \frac{1}{2\left(1+2\lambda^2\theta\right)} \iota + N_3(0)\right] x_t^D(1) \end{aligned} \tag{73}$$

where ι is a 2x1 vector: $(1, -1)'$. The expectation for a Foreign investor is identical.

5.4 Average asset return

We now have expressions for $k_{t+1}^A(1)$, $k_{t+1}^D(1)$, $q_{t+1}^A(1)$, $q_{t+1}^D(1)$ which are the drivers of asset returns. Using the form of the solutions we write the linear component of (29) as:

$$r_{t+1}^A = \left[(1-r_q)(I_2 - \omega I_4) + r_q \alpha_{qA}\right] S_{t+1} + r_q \alpha_{5,qA} x_{t+1}^D - \alpha_{qA} S_t - \alpha_{5,qA} x_t^D$$

Using (58) and (73) we write the first-order expectation by a Home investor as:

$$\begin{aligned} & \left[E_t^{Hj} r_{t+1}^A\right](1) \\ = \; & \left[(1-r_q)(I_2 - \omega I_4) + r_q \alpha_{qA}(0)\right] N_1(0) S_t(1) \\ & + \left[(1-r_q)(I_2 - \omega I_4) + r_q \alpha_{qA}(0)\right] \left[N_2 \frac{1}{2\left(1+2\lambda^2\theta\right)} \iota + N_3(0)\right] x_t^D(1) \\ & - \alpha_{qA} S_t - \alpha_{5,qA} x_t^D \end{aligned}$$

This implies that $\left[E_t^{Hj} r_{t+1}^A\right](1) = \left[E_t^{Fj} r_{t+1}^A\right](1)$.

5.5 Consumption of old agents

The zero-order portfolio shares do not enter our solution so far, which can seem odd as in the presence of incomplete asset markets these shares should matter for the first-order dynamics. Under our specific model this is indeed the case, but the link operates solely through the consumption of old agents. Taking a linear expansion of the budget constraint (14) around the steady state, where $\exp\left[c_o(0)\right] = (1-\bar{c})(\bar{c})^{-1}\exp\left[r(0)+c_y(0)\right]$, we get:

$$c_{o,t+1}^{Hj}(1) = r_{t+1}^{p,Hj}(1) + \frac{1}{1-\bar{c}}\left[a_{Ht}(1) + (1-\omega)k_{H,t}(1)\right] - \frac{\bar{c}}{1-\bar{c}}c_{y,t}^{Hj}(1)$$

A similar expression holds for Foreign investors. Next, we aggregate these relations across investors in each country, and use the linear components of (31)-(32) to write the results in terms of worldwide averages and cross-country differences:

$$\begin{aligned} c_{o,t+1}^{A}(1) &= r_{t+1}^{A}(1) + \frac{1}{1-\bar{c}}\left[a_t^A(1) + (1-\omega)k_t^A(1)\right] - \frac{\bar{c}}{1-\bar{c}}c_{y,t}^{A}(1) \\ c_{o,t+1}^{D}(1) &= z^D(0)\left(r_{H,t+1}(1) - r_{F,t+1}(1)\right) + a_t^D(1) + (1-\omega)k_t^D(1) \end{aligned}$$

where we used the fact that $z^A(0) = 0.5$ and (68). The second relation shows that return differentials lead to unexpected differentials across countries in old agents's consumption in the presence of bias in portfolio holdings ($z^D(0) \neq 0$).

6 Second-order solution

6.1 Worldwide averages of equity prices and capital

Computing the first-order solution for z_t^D from (49) requires the second-order solution of returns, which include asset prices. Our next step is then to derive the second-order solutions for q_t^D, q_t^A, k_t^D and k_t^A, which in turn depend on the second-order solution for c_{yt}^A.

We use the second-order component of (33):

$$\begin{aligned} k_{t+1}^A(2) &= \frac{1}{\xi}q_t^A(2) + k_t^A(2) + \frac{1}{2}\frac{\xi-1}{\xi^2}\left[\left(q_t^A(1)\right)^2 + \frac{1}{4}\left(q_t^D(1)\right)^2\right] \\ &= \frac{1}{\xi}q_t^A(2) + k_t^A(2) + \frac{\xi-1}{2\xi^2}\left[\alpha_{2,qA}(0)a_t^A(1) + \alpha_{4,qA}(0)k_t^A(1)\right]^2 \\ &\quad + \frac{\xi-1}{8\xi^2}\left[\alpha_{1,qD}(0)a_t^D(1) + \alpha_{3,qD}(0)k_t^D(1) + \alpha_{5,qD}(0)x_t^D(1)\right]^2 \end{aligned}$$

We define I_i as a 1x4 vector of zeros with 1 in the i'th position, and define:

$$\begin{aligned} N &= \frac{1}{4}\left[\alpha_{1,qD}(0) I_1 + \alpha_{3,qD}(0) I_3\right]' \left[\alpha_{1,qD}(0) I_1 + \alpha_{3,qD}(0) I_3\right] \\ &\quad + \left[\alpha_{2,qA}(0) I_2 + \alpha_{4,qA}(0) I_4\right]' \left[\alpha_{2,qA}(0) I_2 + \alpha_{4,qA}(0) I_4\right] \end{aligned} \tag{74}$$

Similarly the zero-order coefficients for the worldwide averages of consumption are:

$$\begin{aligned} k_{t+1}^A(2) &= \frac{1}{\xi} q_t^A(2) + k_t^A(2) + \frac{\xi-1}{2\xi^2} S_t'(1) N S_t(1) + \frac{\xi-1}{8\xi^2}\left[\alpha_{5,qD}(0) x_t^D(1)\right]^2 \\ &\quad + \frac{\xi-1}{4\xi^2}\alpha_{5,qD}(0) x_t^D(1)\left[\alpha_{1,qD}(0) a_t^D(1) + \alpha_{3,qD}(0) k_t^D(1)\right] \end{aligned} \tag{75}$$

where we used our first-order results. Similarly, we use the second-order component of (35):

$$\begin{aligned} & q_t^A(2) + k_{t+1}^A(2) + \frac{1}{2}\left[\left(q_t^A(1) + k_{t+1}^A(1)\right)^2 + \frac{1}{4}\left(q_t^D(1) + k_{t+1}^D(1)\right)^2\right] \\ = \; & \frac{1}{1-\bar{c}} a_t^A(2) + \frac{1-\omega}{1-\bar{c}} k_t^A(2) - \frac{\bar{c}}{1-\bar{c}} c_{yt}^A(2) \\ & + \frac{1}{1-\bar{c}}\frac{1}{2}\left[\left[a_t^A(1) + (1-\omega) k_t^A(1)\right]^2 + \frac{1}{4}\left[a_t^D(1) + (1-\omega) k_t^D(1)\right]^2\right] \\ & - \frac{\bar{c}}{1-\bar{c}}\frac{1}{2}\left[\left(c_{yt}^A(1)\right)^2 + \frac{1}{4}\left(c_{yt}^D(1)\right)^2\right] - \frac{\bar{c}}{1-\bar{c}}\frac{1}{4}\left[D_t^H(c)(2) + D_t^F(c)(2)\right] \end{aligned}$$

A linear approximation of (15) shows that the first-order consumption of young agents is the same across agents:

$$\frac{\gamma}{1-\bar{c}}\left[a_{Ht}(1) + (1-\omega) k_{H,t}(1) - c_{y,t}^{Hj}(1)\right] = (1-\gamma)\left[E_t^{Hj} r_{t+1}^{p,Hj}\right](1)$$

Agents have the same first-order expectations, as shown in the signal extraction. Furthermore the first-order expected excess returns are zero, so any heterogeneity in zero-order portfolio shares does not translate into an heterogeneity in expected portfolio return. As the first-order consumption of young agents is the same for all individuals within a country, we get $D_t^H(c)(2) = D_t^F(c)(2) = 0$. Using our first order solution, the second-order component of (35) is re-written as:

$$\begin{aligned}
& q_t^A(2) + k_{t+1}^A(2) \\
= \; & \frac{1}{1-\bar{c}} a_t^A(2) + \frac{1-\omega}{1-\bar{c}} k_t^A(2) \\
& -\frac{\bar{c}}{1-\bar{c}} \left[\begin{array}{c} \alpha_{cA}(1) S_t(1) + \alpha_{cA}(0) S_t(2) + \alpha_{5,cA}(1) x_t^D(1) \\ +S_t'(1) A_{cA}(0) S_t(1) + \beta_{cA}(0) S_t(1) x_t^D(1) \\ +\mu_{cA}(0) \left(x_t^D(1)\right)^2 + \kappa_{cA}(2) \end{array} \right] \\
& -\frac{1}{2\bar{c}} \left[\left[1 - (1+\xi) \alpha_{2,kA}(0)\right] a_t^A(1) + \left[(1+\xi)(1-\alpha_{4,kA}(0)) - \omega\right] k_t^A(1) \right]^2 \\
& +\frac{1}{8} \left[a_t^D(1) + (1-\omega) k_t^D(1) \right]^2 \\
& -\frac{1}{8} \left[\frac{1+\xi}{\xi} \alpha_{1,qD}(0) a_t^D(1) + \left[1 + \frac{1+\xi}{\xi} \alpha_{3,qD}(0)\right] k_t^D(1) + \frac{1+\xi}{\xi} \alpha_{5,qD}(0) x_t^D(1) \right]^2
\end{aligned}$$

where we used the form of c_{yt}^A to write:

$$\begin{aligned}
c_{yt}^A(2) \; = \; & \alpha_{cA}(1) S_t(1) + \alpha_{cA}(0) S_t(2) + \alpha_{5,cA}(1) x_t^D(1) \\
& +S_t'(1) A_{cA}(0) S_t(1) + \beta_{cA}(0) S_t(1) x_t^D(1) + \mu_{cA}(0) \left(x_t^D(1)\right)^2 + \kappa_{cA}(2)
\end{aligned}$$

Using (74) to substitute for $k_{t+1}^A(2)$, we obtain:

$$\begin{aligned}
& q_t^A(2) \\
= \; & \frac{\xi}{1+\xi} \frac{1}{1-\bar{c}} a_t^A(2) + \frac{\xi}{1+\xi} \frac{\bar{c}-\omega}{1-\bar{c}} k_t^A(2) - \frac{\xi}{1+\xi} \frac{\bar{c}}{1-\bar{c}} \alpha_{cA}(0) S_t(2) \\
& -\frac{\xi}{1+\xi} \frac{\bar{c}}{1-\bar{c}} \left[\alpha_{cA}(1) S_t(1) + \kappa_{cA}(2)\right] \\
& +\frac{1}{8} \frac{\xi}{1+\xi} S_t'(1) \left[\begin{array}{c} -\left[G_{qA,1}\right]' G_{qA,1} \\ +\left[I_1 + (1-\omega) I_3\right]' \left[I_1 + (1-\omega) I_3\right] \\ -\frac{4}{\bar{c}} \left[G_{qA,2}\right]' G_{qA,2} - \frac{\xi-1}{4\xi^2} N - 8 \frac{\bar{c}}{1-\bar{c}} A_{cA}(0) \end{array} \right] S_t(1) \\
& -\frac{\xi}{1+\xi} \left[\frac{1}{8} \frac{\xi+3}{\xi} \left[\alpha_{5,qD}(0)\right]^2 + \frac{\bar{c}}{1-\bar{c}} \mu_{cA}(0) \right] \left[x_t^D(1)\right]^2 \\
& -\frac{\xi}{1+\xi} \frac{\bar{c}}{1-\bar{c}} \alpha_{5,cA}(1) x_t^D(1) \\
& -\frac{\xi}{1+\xi} \left[\frac{1}{4} \left[\begin{array}{c} \frac{3+\xi}{\xi} \alpha_{1,qD}(0) I_1 \\ + \left[\frac{1+\xi}{\xi} + \frac{3+\xi}{\xi} \alpha_{3,qD}(0)\right] I_3 \end{array} \right] \alpha_{5,qD}(0) + \frac{\bar{c}}{1-\bar{c}} \beta_{cA}(0) \right] S_t(1) x_t^D(1)
\end{aligned}$$

where:

$$
\begin{aligned}
G_{qA,1} &= \frac{1+\xi}{\xi}\alpha_{1,qD}(0)\, I_1 + \left[1+\frac{1+\xi}{\xi}\alpha_{3,qD}(0)\right] I_3 \\
G_{qA,2} &= \left[1-(1+\xi)\,\alpha_{2,kA}(0)\right] I_2 + \left[(1+\xi)\,(1-\alpha_{4,kA}(0))-\omega\right] I_4
\end{aligned}
$$

From our conjecture on the form of q_t^A we have:

$$
\begin{aligned}
q_t^A(2) &= \alpha_{qA}(1)\, S_t(1) + \alpha_{qA}(0)\, S_t(2) + \alpha_{5,qA}(1)\, x_t^D(1) \\
&\quad + S_t'(1)\, A_{qA}(0)\, S_t(1) + \beta_{qA}(0)\, S_t(1)\, x_t^D(1) + \mu_{qA}(0)\left(x_t^D(1)\right)^2 + \kappa_{qA}(2)
\end{aligned}
$$

Equalizing coefficients across the two relations, we get:

$$
\begin{aligned}
\kappa_{qA}(2) &= -\frac{\xi}{1+\xi}\frac{\bar{c}}{1-\bar{c}}\kappa_{cA}(2) \\
\alpha_{5,qA}(1) &= -\frac{\xi}{1+\xi}\frac{\bar{c}}{1-\bar{c}}\alpha_{5,cA}(1) \\
\alpha_{qA}(1) &= -\frac{\xi}{1+\xi}\frac{\bar{c}}{1-\bar{c}}\alpha_{cA}(1) \\
\alpha_{qA}(0) &= \frac{\xi}{1+\xi}\frac{1}{1-\bar{c}}I_2 + \frac{\xi}{1+\xi}\frac{\bar{c}-\omega}{1-\bar{c}}I_4 - \frac{\xi}{1+\xi}\frac{\bar{c}}{1-\bar{c}}\alpha_{cA}(0) \\
\beta_{qA}(0) &= -\frac{\xi}{1+\xi}\frac{1}{4}\left[\frac{3+\xi}{\xi}\alpha_{1,qD}(0)\, I_1 + \left[\frac{1+\xi}{\xi}+\frac{3+\xi}{\xi}\alpha_{3,qD}(0)\right] I_3\right]\alpha_{5,qD}(0) \\
&\quad -\frac{\xi}{1+\xi}\frac{\bar{c}}{1-\bar{c}}\beta_{cA}(0) \\
\mu_{qA}(0) &= -\frac{\xi}{1+\xi}\left[\frac{1}{8}\frac{\xi+3}{\xi}\left[\alpha_{5,qD}(0)\right]^2 + \frac{\bar{c}}{1-\bar{c}}\mu_{cA}(0)\right] \\
A_{qA}(0) &= \frac{1}{8}\frac{\xi}{1+\xi}\left[\begin{array}{c} -\left[\begin{array}{c}\frac{1+\xi}{\xi}\alpha_{1,qD}(0)\, I_1 \\ +\left[1+\frac{1+\xi}{\xi}\alpha_{3,qD}(0)\right] I_3\end{array}\right]'\left[\begin{array}{c}\frac{1+\xi}{\xi}\alpha_{1,qD}(0)\, I_1 \\ +\left[1+\frac{1+\xi}{\xi}\alpha_{3,qD}(0)\right] I_3\end{array}\right] \\ +\left[I_1+(1-\omega)\, I_3\right]'\left[I_1+(1-\omega)\, I_3\right] \\ -\frac{\xi-1}{4\xi^2}N - 8\frac{\bar{c}}{1-\bar{c}}A_{cA}(0)\end{array}\right] \\
&\quad -\frac{1}{2\bar{c}}\frac{\xi}{1+\xi}\left[\begin{array}{c}\left[1-(1+\xi)\,\alpha_{2,kA}(0)\right] I_2 \\ +\left[(1+\xi)\,(1-\alpha_{4,kA}(0))-\omega\right] I_4\end{array}\right]'\left[\begin{array}{c}\left[1-(1+\xi)\,\alpha_{2,kA}(0)\right] I_2 \\ +\left[(1+\xi)\,(1-\alpha_{4,kA}(0))-\omega\right] I_4\end{array}\right]
\end{aligned}
\tag{76}
$$

The restriction on $\alpha_{qA}(0)$ leads to the same coefficients as in the first-order solution above.

From our conjecture on the form of k_t^A we have:

$$
\begin{aligned}
k_t^A(2) &= \alpha_{kA}(1)\, S_t(1) + \alpha_{kA}(0)\, S_t(2) + \alpha_{5,kA}(1)\, x_t^D(1) \\
&\quad + S_t'(1)\, A_{kA}(0)\, S_t(1) + \beta_{kA}(0)\, S_t(1)\, x_t^D(1) + \mu_{kA}(0)\left(x_t^D(1)\right)^2 + \kappa_{kA}(2)
\end{aligned}
$$

Using (75) and equalizing coefficients we get:

$$\begin{aligned}
\kappa_{kA}(2) &= \frac{1}{\xi}\kappa_{qA}(2) \\
\alpha_{5,kA}(1) &= \frac{1}{\xi}\alpha_{5,qA}(1) \\
\alpha_{kA}(1) &= \frac{1}{\xi}\alpha_{qA}(1) \\
\alpha_{kA}(0) &= I_4 + \frac{1}{\xi}\alpha_{qA}(0) \\
\beta_{kA}(0) &= \frac{1}{\xi}\beta_{qA}(0) + \frac{\xi-1}{4\xi^2}\alpha_{5,qD}(0)\left[\alpha_{1,qD}(0)I_1 + \alpha_{3,qD}(0)I_3\right] \\
\mu_{kA}(0) &= \frac{1}{\xi}\mu_{qA}(0) + \frac{\xi-1}{8\xi^2}\left[\alpha_{5,qD}(0)\right]^2 \\
A_{kA}(0) &= \frac{1}{\xi}A_{qA}(0) + \frac{\xi-1}{2\xi^2}N
\end{aligned} \tag{77}$$

The solution for $\alpha_{kA}(0)$ is the same as derived in the previous section based on the first-order solution. At this point we have solved for the coefficients on $k_{t+1}^{A}(2)$ and $q_t^A(2)$ conditional on the coefficients on $c_{yt}^{A}(2)$.

6.2 Cross-country differences of equity prices and capital

From our conjecture on the form of q_t^D we have:

$$\begin{aligned}
q_t^D(2) &= \alpha_{qD}(1)S_t(1) + \alpha_{qD}(0)S_t(2) + \alpha_{5,qD}(1)x_t^D(1) \\
&\quad + S_t'(1)A_{qD}(0)S_t(1) + \beta_{qD}(0)S_t(1)x_t^D(1) + \mu_{qD}(0)\left(x_t^D(1)\right)^2 + \kappa_{qD}
\end{aligned}$$

We use the second-order component of (34):

$$
\begin{aligned}
k_{t+1}^{D}(2) &= k_t^D(2) + \frac{1}{\xi} q_t^D(2) + \frac{\xi-1}{\xi^2} q_t^D(1)\, q_t^A(1) \\
&= k_t^D(2) + \frac{1}{\xi} q_t^D(2) \\
&\quad + \frac{\xi-1}{\xi^2} S_t(1)' \begin{bmatrix} \alpha_{1,qD}(0)\, I_1 \\ +\alpha_{3,qD}(0)\, I_3 \end{bmatrix}' \begin{bmatrix} \xi\alpha_{2,kA}(0)\, I_2 \\ +\xi\left(\alpha_{4,kA}(0)-1\right) I_4 \end{bmatrix} S_t(1) \\
&\quad + \frac{\xi-1}{\xi^2} \alpha_{5,qD}(0) \left[\xi\alpha_{2,kA}(0)\, I_2 + \xi\left(\alpha_{4,kA}(0)-1\right) I_4\right] S_t(1) x_t^D(1) \\
&= k_t^D(2) + \frac{\xi-1}{\xi^2} S_t(1)' \begin{bmatrix} \alpha_{1,qD}(0)\, I_1 \\ +\alpha_{3,qD}(0)\, I_3 \end{bmatrix}' \begin{bmatrix} \xi\alpha_{2,kA}(0)\, I_2 \\ +\xi\left(\alpha_{4,kA}(0)-1\right) I_4 \end{bmatrix} S_t(1) \\
&\quad + \frac{\xi-1}{\xi^2} \alpha_{5,qD}(0) \left[\xi\alpha_{2,kA}(0)\, I_2 + \xi\left(\alpha_{4,kA}(0)-1\right) I_4\right] S_t(1) x_t^D(1) \\
&\quad + \frac{1}{\xi}\alpha_{qD}(1)\, S_t(1) + \frac{1}{\xi}\alpha_{qD}(0)\, S_t(2) + \frac{1}{\xi}\alpha_{5,qD}(1)\, x_t^D(1) \\
&\quad + \frac{1}{\xi} S_t'(1)\, A_{qD}(0)\, S_t(1) + \frac{1}{\xi}\beta_{qD}(0)\, S_t(1)\, x_t^D(1) \\
&\quad + \frac{1}{\xi}\mu_{qD}(0) \left(x_t^D(1)\right)^2 + \frac{1}{\xi}\kappa_{qD}
\end{aligned}
$$

From our conjecture on the form of k_{t+1}^D we have:

$$
\begin{aligned}
k_{t+1}^{D}(2) &= \alpha_{kD}(1)\, S_t(1) + \alpha_{kD}(0)\, S_t(2) + \alpha_{5,kD}(1)\, x_t^D(1) \\
&\quad + S_t'(1)\, A_{kD}(0)\, S_t(1) + \beta_{kD}(0)\, S_t(1)\, x_t^D(1) + \mu_{kD}(0)\left(x_t^D(1)\right)^2 + \kappa_{kD}(2)
\end{aligned}
$$

Equalizing coefficients we get:

$$\begin{aligned}
\kappa_{kD}(2) &= \frac{1}{\xi}\kappa_{qD}(2) \\
\alpha_{5,kD}(1) &= \frac{1}{\xi}\alpha_{5,qD}(1) \\
\alpha_{kD}(1) &= \frac{1}{\xi}\alpha_{qD}(1) \\
\alpha_{kD}(0) &= I_3 + \frac{1}{\xi}\alpha_{qD}(0) \\
\beta_{kD}(0) &= \frac{\xi-1}{\xi}\alpha_{5,qD}(0)\left[\alpha_{2,kA}(0)I_2 + (\alpha_{4,kA}(0)-1)I_4\right] + \frac{1}{\xi}\beta_{qD}(0) \\
\mu_{kD}(0) &= \frac{1}{\xi}\mu_{qD}(0) \\
A_{kD}(0) &= \frac{\xi-1}{\xi}\begin{bmatrix} \alpha_{1,qD}(0)I_1 \\ +\alpha_{3,qD}(0)I_3 \end{bmatrix}'\begin{bmatrix} \alpha_{2,kA}(0)I_2 \\ +(\alpha_{4,kA}(0)-1)I_4 \end{bmatrix} + \frac{1}{\xi}A_{qD}(0)
\end{aligned} \tag{78}$$

The solution for $\alpha_{kD}(0)$ is the same as derived in the previous section based on the first-order solution.

6.3 Expectations of state variables

Using (57), the second-order component of the expectation of state variables by a Home investor j is:

$$\begin{aligned}
&\left[E_t^{Hj}S_{t+1}\right](2) \\
= \; & N_1(0)S_t(2) + N_1(1)S_t(1) + N_3(1)x_t^D(1) \\
&+N_4(0)\left(x_t^D(1)\right)^2 + N_5(0)S_t(1)x_t^D(1) + \begin{pmatrix} 0 \\ 0 \\ S_t'(1)N_6(0)S_t(1) \\ S_t'(1)N_7(0)S_t(1) \end{pmatrix} + \kappa(2) \\
&+N_2(0)\left[E_t^{Hj}\varepsilon_{t+1}\right](2)
\end{aligned} \tag{79}$$

recalling that N_2 only has zero-order components. Using the results of the signal extraction, we get:

$$\left[E_t^{Hj}\varepsilon_{t+1}\right](2) = \frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)}\begin{vmatrix} \frac{1+4\lambda^2\theta}{\sigma_{H,H}^2}\epsilon_{j,t}^{H,H} + \frac{1}{\sigma_{H,F}^2}\epsilon_{j,t}^{H,F} \\ \frac{1}{\sigma_{H,H}^2}\epsilon_{j,t}^{H,H} + \frac{1+4\lambda^2\theta}{\sigma_{H,F}^2}\epsilon_{j,t}^{H,F} \end{vmatrix} \tag{80}$$

Aggregating across agents, we get:

$$\left[\bar{E}_t^H \varepsilon_{t+1}\right](2) = \int \left[E_t^{Hj} \varepsilon_{t+1}\right](2)\, dj = 0_{2x1} = \left[\bar{E}_t^H \varepsilon_{t+1}\right](2)$$

Therefore $\left[\bar{E}_t^H S_{t+1}\right](2) = \left[\bar{E}_t^F S_{t+1}\right](2)$.

Another useful result is to compute the following expectation, for some 4x4 matrix B with only zero-order elements:

$$\begin{aligned}
& \left[E_t^{Hj} S_{t+1}' B S_{t+1}\right](2) \\
= & \left[E_t^{Hj} \begin{bmatrix} N_1(0) S_t + N_3(0) x_t^D(1) \\ +N_2 \varepsilon_{t+1} \end{bmatrix}' B \begin{bmatrix} N_1(0) S_t + N_3(0) x_t^D(1) \\ +N_2 \varepsilon_{t+1} \end{bmatrix}\right](2) \\
= & \begin{bmatrix} N_1(0) S_t(1) \\ +N_3(0) x_t^D(1) \end{bmatrix}' B \begin{bmatrix} N_1(0) S_t(1) \\ +N_3(0) x_t^D(1) \end{bmatrix} \\
& + \begin{bmatrix} N_1(0) S_t(1) \\ +N_3(0) x_t^D(1) \end{bmatrix}' B N_2 \left[E_t^{Hj} \varepsilon_{t+1}\right](1) \\
& + \left[E_t^{Hj} \varepsilon_{t+1}'\right](1) N_2' B \begin{bmatrix} N_1(0) S_t(1) \\ +N_3(0) x_t^D(1) \end{bmatrix} \\
& + \left[E_t^{Hj} \varepsilon_{t+1}' N_2' B N_2 \varepsilon_{t+1}\right](2)
\end{aligned}$$

Using results from the signal extraction, we write:

$$\left[E_t^{Hj} \varepsilon_{t+1}\right](1) = \frac{1}{2\left(1 + 2\lambda^2 \theta\right)} \iota x_t^D(1) = \left[E_t^{Hj} \varepsilon_{t+1}'\right](1)'$$

where ι is a 2x1 vector: $(1, -1)'$. In addition:

$$\begin{aligned}
& \left[E_t^{Hj} \varepsilon_{t+1}' N_2' B N_2 \varepsilon_{t+1}\right](2) \\
= & \left[E_t^{Hj} \varepsilon_{t+1}' \begin{matrix} \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \end{matrix} \varepsilon_{t+1}\right](2) \\
= & \left(\tilde{B}_{11} + \tilde{B}_{22}\right) \left[E_t^{Hj} \left(\varepsilon_{H,t+1}\right)^2\right](2) + \left(\tilde{B}_{12} + \tilde{B}_{21}\right) \left[E_t^{Hj} \left(\varepsilon_{H,t+1} \varepsilon_{F,t+1}\right)\right](2) \\
= & \left(\tilde{B}_{11} + \tilde{B}_{22}\right) \left[\frac{1 + 4\lambda^2 \theta}{2\left(1 + 2\lambda^2 \theta\right)} \sigma_a^2 + \left(\frac{1}{2\left(1 + 2\lambda^2 \theta\right)} x_t^D(1)\right)^2\right] \\
& + \left(\tilde{B}_{12} + \tilde{B}_{21}\right) \left[\frac{1}{2\left(1 + 2\lambda^2 \theta\right)} \sigma_a^2 - \left(\frac{1}{2\left(1 + 2\lambda^2 \theta\right)} x_t^D(1)\right)^2\right]
\end{aligned}$$

where $\tilde{B}$ is a 2x2 matrix equal to $N_2' B N_2$. Therefore we get:

$$\begin{aligned}
& \left[E_t^{Hj} S_{t+1}' B S_{t+1}\right](2) \\
= \; & S_t(1)' N_1(0)' B N_1(0) S_t(1) \\
& + \left[N_3(0)' B N_3(0) + \frac{1}{2\left(1+2\lambda^2\theta\right)} N_3(0)' \left(B + B'\right) N_2 \iota\right] \left(x_t^D(1)\right)^2 \\
& + S_t(1)' N_1(0)' \left(B + B'\right) \left[N_3(0) + \frac{1}{2\left(1+2\lambda^2\theta\right)} N_2 \iota\right] x_t^D(1) \\
& + \left(\tilde{B}_{11} + \tilde{B}_{22}\right) \left[\frac{1+4\lambda^2\theta}{2\left(1+2\lambda^2\theta\right)} \sigma_a^2 + \left(\frac{1}{2\left(1+2\lambda^2\theta\right)} x_t^D(1)\right)^2\right] \\
& + \left(\tilde{B}_{12} + \tilde{B}_{21}\right) \left[\frac{1}{2\left(1+2\lambda^2\theta\right)} \sigma_a^2 - \left(\frac{1}{2\left(1+2\lambda^2\theta\right)} x_t^D(1)\right)^2\right]
\end{aligned} \tag{81}$$

Notice that the expression is the same for any agent in either country.

6.4 Average asset return

We now have expressions for $k_{t+1}^A(2)$, $k_{t+1}^D(2)$, $q_{t+1}^A(2)$, $q_{t+1}^D(2)$ which are the drivers of asset returns. Using the form of the solutions we write (29) as:

$$\begin{aligned}
r_{t+1}^A = \; & \left[\left(1-r_q\right)\left(I_2 - \omega I_4\right) + r_q \alpha_{qA}\right] S_{t+1} \\
& + r_q \left[\alpha_{5,qA} x_{t+1}^D + S_{t+1}' A_{qA} S_{t+1} + \beta_{qA} S_{t+1} x_{t+1}^D + \mu_{qA}\left(x_{t+1}^D\right)^2\right] \\
& - \alpha_{qA} S_t - \alpha_{5,qA} x_t^D - S_t' A_{qA} S_t - \beta_{qA} S_t x_t^D - \mu_{qA}\left(x_t^D\right)^2 - \left(1-r_q\right)\kappa_{qA} \\
& + \frac{r_q\left(1-r_q\right)}{2} \left[\left(-I_2 + \omega I_4 + \alpha_{qA}\right) S_{t+1} + \alpha_{5,qA} x_{t+1}^D\right]^2 \\
& + \frac{r_q\left(1-r_q\right)}{8} \left[\left(-I_1 + \omega I_3 + \alpha_{qD}\right) S_{t+1} + \alpha_{5,qD} x_{t+1}^D\right]^2
\end{aligned}$$

which we rewrite as:

$$\begin{aligned}
r_{t+1}^A = \; & -\alpha_{qA} S_t - \alpha_{5,qA} x_t^D - S_t' A_{qA} S_t - \beta_{qA} S_t x_t^D - \mu_{qA}\left(x_t^D\right)^2 - \left(1-r_q\right)\kappa_{qA} \\
& + \left[\left(1-r_q\right)\left(I_2 - \omega I_4\right) + r_q \alpha_{qA}\right] S_{t+1} + r_q \alpha_{5,qA} x_{t+1}^D + S_{t+1}' \Phi_{ra1} S_{t+1} \\
& + \left[r_q \mu_{qA} + \frac{r_q\left(1-r_q\right)}{2}\left(\alpha_{5,qA}\right)^2 + \frac{r_q\left(1-r_q\right)}{8}\left(\alpha_{5,qD}\right)^2\right]\left(x_{t+1}^D\right)^2 \\
& + \left[\begin{array}{c} r_q \beta_{qA} + r_q\left(1-r_q\right)\left[\alpha_{qA} - \left(I_2 - \omega I_4\right)\right]\alpha_{5,qA} \\ + \frac{r_q(1-r_q)}{4}\left[\alpha_{qD} - \left(I_1 - \omega I_3\right)\right]\alpha_{5,qD} \end{array}\right] S_{t+1} x_{t+1}^D
\end{aligned}$$

where:

$$\Phi_{ra1} = r_q A_{qA} + \frac{r_q(1-r_q)}{2}\left[\begin{array}{c} [\alpha_{qA}-(I_2-\omega I_4)]'[\alpha_{qA}-(I_2-\omega I_4)] \\ +\frac{1}{4}[\alpha_{qD}-(I_1-\omega I_3)]'[\alpha_{qD}-(I_1-\omega I_3)] \end{array}\right] \tag{82}$$

We not turn to expectations. First, notice that x^D_{t+1} and S_{t+1} are orthogonal, so $\left[E^{Hj}_t S_{t+1}x^D_{t+1}\right](2) = 0$. Using (58) and (59), as well as our results for zero-order coefficients (namely $\alpha_{5,qA}(0) = 0$), the second-order component of a Home investor's expected average return is:

$$\begin{aligned} & \left[E^{Hj}_t r^A_{t+1}\right](2) \\ = \; & -\alpha_{qA}(1)S_t(1) - \alpha_{qA}(0)S_t(2) - S'_t(1)A_{qA}(0)S_t(1) - \alpha_{5,qA}(1)x^D_t(1) \\ & -\beta_{qA}(0)S_t(1)x^D_t(1) - \mu_{qA}(0)\left(x^D_t(1)\right)^2 - (1-r_q)\kappa_{qA} \\ & +[(1-r_q)(I_2-\omega I_4)+r_q\alpha_{qA}(0)]\left[E^{Hj}_t S_{t+1}\right](2) + r_q\alpha_{qA}(1)\left[E^{Hj}_t S_{t+1}\right](1) \\ & +\left[E^{Hj}_t S'_{t+1}(1)\Phi_{ra1}S_{t+1}(1)\right](2) \\ & +\left[r_q\mu_{qA}(0) + \frac{r_q(1-r_q)}{8}(\alpha_{5,qD}(0))^2\right]2\sigma^2_a\left[1+\lambda^2\theta(1-\rho_\tau)\right] \end{aligned} \tag{83}$$

where $\left[E^{Hj}_t S'_{t+1}(1)\Phi_{ra1}S_{t+1}(1)\right](2)$ can be computed using (81). Recall that $\left[E^{Hj}_t S_{t+1}\right](1)$ is the same for all agents in both countries, and $\left[\bar{E}^H_t S_{t+1}\right](2) = \left[\bar{E}^F_t S_{t+1}\right](2)$. This implies that once we aggregate across agents in each country:

$$\left[\bar{E}^H_t r^A_{t+1}\right](2) = \left[\bar{E}^F_t r^A_{t+1}\right](2) = \left[\bar{E}_t r^A_{t+1}\right](2) \tag{84}$$

(83) can be split into the common components and the idiosyncratic ones, which only enter through $\left[E^{Hj}_t S_{t+1}\right](2)$. Using (79) and the results from the signal extraction, we get:

$$\begin{aligned} & \left[E^{Hj}_t r^A_{t+1}\right](2) \\ = \; & \left[\bar{E}_t r^A_{t+1}\right](2) + [(1-r_q)(I_2-\omega I_4)+r_q\alpha_{qA}(0)]N_2(0)\left[E^{Hj}_t \varepsilon_{t+1}\right](2) \\ = \; & \left[\bar{E}_t r^A_{t+1}\right](2) + [(1-r_q)(I_2-\omega I_4)+r_q\alpha_{qA}(0)]\left|\begin{array}{c} \frac{2\lambda^2\theta}{1+2\lambda^2\theta}\left(\frac{\epsilon^{H,H}_{j,t}}{\sigma^2_{H,H}} - \frac{\epsilon^{H,F}_{j,t}}{\sigma^2_{H,F}}\right) \\ \frac{1}{2}\left(\frac{\epsilon^{H,H}_{j,t}}{\sigma^2_{H,H}} + \frac{\epsilon^{H,F}_{j,t}}{\sigma^2_{H,F}}\right) \\ 0 \\ 0 \end{array}\right|\sigma^2_a \\ = \; & \left[\bar{E}_t r^A_{t+1}\right](2) + \Theta_{z,ra}\left(\frac{\epsilon^{H,H}_{j,t}}{\sigma^2_{H,H}} + \frac{\epsilon^{H,F}_{j,t}}{\sigma^2_{H,F}}\right) \end{aligned} \tag{85}$$

where the second order coefficient $\Theta_{z,ra}$ is:

$$\Theta_{z,ra} = [1 - r_q + r_q \alpha_{2,qA}(0)] \frac{\sigma_a^2}{2}$$

Following similar steps, the expectation of a Foreign investor is:

$$\left[E_t^{Fj} r_{t+1}^A\right](2) = \left[\bar{E}_t r_{t+1}^A\right](2) + \Theta_{z,ra}\left(\frac{\epsilon_{j,t}^{F,H}}{\sigma_{H,F}^2} + \frac{\epsilon_{j,t}^{F,F}}{\sigma_{H,H}^2}\right) \tag{86}$$

We can also compute the expectation of the squared return (recalling that $\alpha_{5,qA}(0) = 0$):

$$\begin{aligned}
&\left[E_t^{Hj}\left(r_{t+1}^A\right)^2\right](2) \\
= \;& S_t'(1)\,\alpha_{qA}'(0)\,\alpha_{qA}(0)\,S_t(1) + \left[E_t^{Hj} S_{t+1}'(1)\,\Phi_{ra2} S_{t+1}(1)\right](2) \\
&-2S_t'(1)\,\alpha_{qA}'(0)\begin{bmatrix}(1-r_q)(I_2-\omega I_4) \\ +r_q\alpha_{qA}(0)\end{bmatrix} N_1(0)\,S_t(1) \\
&-2S_t'(1)\,\alpha_{qA}'(0)\begin{bmatrix}(1-r_q)(I_2-\omega I_4) \\ +r_q\alpha_{qA}(0)\end{bmatrix}\left[N_2\frac{1}{2\left(1+2\lambda^2\theta\right)}\iota + N_3(0)\right] x_t^D(1)
\end{aligned} \tag{87}$$

where:

$$\Phi_{ra2} = \begin{bmatrix}(1-r_q)(I_2-\omega I_4) \\ +r_q\alpha_{qA}\end{bmatrix}'\begin{bmatrix}(1-r_q)(I_2-\omega I_4) \\ +r_q\alpha_{qA}\end{bmatrix} \tag{88}$$

and we used (73). Notice that $\left[E_t^{Hj}\left(r_{t+1}^A\right)^2\right](2) = \left[E_t^{Fj}\left(r_{t+1}^A\right)^2\right](2) = \left[E_t\left(r_{t+1}^A\right)^2\right](2)$.

6.5 Excess return on asset

Using our solution we write (30) as:

$$\begin{aligned}
er_{t+1} \;=\; & -\alpha_{qD}S_t - \alpha_{5,qD}x_t^D - S_t'A_{qD}S_t - \beta_{qD}S_t x_t^D - \mu_{qD}\left(x_t^D\right)^2 \\
&+\left[(1-r_q)(I_1-\omega I_3) + r_q\alpha_{qD}\right]S_{t+1} + r_q\alpha_{5,qD}x_{t+1}^D - (1-r_q)\kappa_{qD} \\
&+S_{t+1}'\Phi_{er1}S_{t+1} + \left[r_q\mu_{qD} + r_q(1-r_q)\alpha_{5,qA}\alpha_{5,qD}\right]\left(x_{t+1}^D\right)^2 \\
&+\left[r_q\beta_{qD} + r_q(1-r_q)\begin{bmatrix}\alpha_{5,qA}\left[\alpha_{qD}-(I_1-\omega I_3)\right] \\ +\alpha_{5,qD}\left[\alpha_{qA}-(I_2-\omega I_4)\right]\end{bmatrix}\right]S_{t+1}x_{t+1}^D
\end{aligned}$$

where:

$$\Phi_{er1} = r_qA_{qD} + r_q(1-r_q)\left[\alpha_{qA}-(I_2-\omega I_4)\right]'\left[\alpha_{qD}-(I_1-\omega I_3)\right] \tag{89}$$

Turning to expectations, the second-order component of a Home investor's expected excess return is:

$$\begin{aligned}
& \left[E_t^{Hj} er_{t+1}\right](2) \\
= \; & -\alpha_{qD}(1) S_t(1) - \alpha_{qD}(0) S_t(2) - S_t'(1) A_{qD}(0) S_t(1) - \alpha_{5,qD}(1) x_t^D(1) \\
& -\beta_{qD}(0) S_t(1) x_t^D(1) - \mu_{qD}(0)\left(x_t^D(1)\right)^2 - (1-r_q)\kappa_{qD} \\
& +\left[(1-r_q)(I_1 - \omega I_3) + r_q \alpha_{qD}(0)\right]\left[E_t^{Hj} S_{t+1}\right](2) + r_q \alpha_{qD}(1)\left[E_t^{Hj} S_{t+1}\right](1) \\
& +\left[E_t^{Hj} S_{t+1}'(1) \Phi_{er1} S_{t+1}(1)\right](2) + r_q \mu_{qD} 2\sigma_a^2\left(1+2\lambda^2\theta\right)
\end{aligned} \tag{90}$$

Aggregating across Home investors and using (73), (79) and (81) we get:

$$\begin{aligned}
& \left[\bar{E}_t^H er_{t+1}\right](2) \\
= \; & er(2) + \alpha_{er}(0) S_t(2) + \alpha_{er}(1) S_t(1) + S_t'(1) A_{er}(0) S_t(1) \\
& +\alpha_{5,er}(1) x_t^D(1) + \beta_{er}(0) S_t(1) x_t^D(1) + \mu_{er}(0)\left(x_t^D(1)\right)^2
\end{aligned} \tag{91}$$

where:

$$
\begin{aligned}
er\left(2\right) &= r_q\mu_{qD}2\sigma_a^2\left[1+\lambda^2\theta\left(1-\rho_\tau\right)\right]-\left(1-r_q\right)\kappa_{qD}+\left[r_q\alpha_{3,qD}\left(0\right)-\omega\left(1-r_q\right)\right]\kappa_{kD} \\
&+\left(\left[\tilde{\Phi}_{er1}\right]_{11}+\left[\tilde{\Phi}_{er1}\right]_{22}\right)\frac{1+4\lambda^2\theta1-\rho_\tau}{2\left(1+2\lambda^2\theta\right)}\sigma_a^2 \\
&+\left(\left[\tilde{\Phi}_{er1}\right]_{12}+\left[\tilde{\Phi}_{er1}\right]_{21}\right)\frac{1}{2\left(1+2\lambda^2\theta\right)}\sigma_a^2 \\
\alpha_{er}\left(0\right) &= -\alpha_{qD}\left(0\right)+\left(1-r_q\right)\left(\rho I_1-\omega\alpha_{kD}\left(0\right)\right)+r_q\alpha_{qD}\left(0\right)N_1\left(0\right) \\
\alpha_{er}\left(1\right) &= -\alpha_{qD}\left(1\right)+r_q\alpha_{qD}\left(1\right)N_1\left(0\right)-\omega\left(1-r_q\right)\alpha_{kD}\left(1\right)+r_q\alpha_{qD}\left(0\right)N_1\left(1\right) \qquad (92) \\
\alpha_{5,er}\left(1\right) &= -\alpha_{5,qD}\left(1\right)+r_q\alpha_{qD}\left(1\right)\left[N_2\frac{1}{2\left(1+2\lambda^2\theta\right)}\iota+N_3\left(0\right)\right] \\
&-\omega\left(1-r_q\right)\alpha_{5,kD}\left(1\right)+r_q\alpha_{qD}\left(0\right)N_3\left(1\right) \\
A_{er}\left(0\right) &= -A_{qD}\left(0\right)+N_1\left(0\right)'\Phi_{er1}N_1\left(0\right)-\omega\left(1-r_q\right)A_{kD}\left(0\right)+r_q\alpha_{3,qD}\left(0\right)A_{kD}\left(0\right) \\
\beta_{er}\left(0\right) &= -\beta_{qD}\left(0\right)-\omega\left(1-r_q\right)\beta_{kD}\left(0\right)+r_q\alpha_{qD}\left(0\right)N_5\left(0\right) \\
&+\left[N_3\left(0\right)+\frac{1}{2\left(1+2\lambda^2\theta\right)}N_2\iota\right]'\left(\Phi_{er1}+\Phi_{er1}'\right)N_1\left(0\right) \\
\mu_{er}\left(0\right) &= -\mu_{qD}\left(0\right)-\omega\left(1-r_q\right)\mu_{kD}\left(0\right)+r_q\alpha_{qD}\left(0\right)N_4\left(0\right) \\
&+\frac{\left[\tilde{\Phi}_{er1}\right]_{11}+\left[\tilde{\Phi}_{er1}\right]_{22}-\left[\tilde{\Phi}_{er1}\right]_{12}-\left[\tilde{\Phi}_{er1}\right]_{21}}{4\left(1+2\lambda^2\theta\right)^2} \\
&+N_3\left(0\right)'\Phi_{er1}N_3\left(0\right)+\frac{1}{2\left(1+2\lambda^2\theta\right)}N_3\left(0\right)'\left(\Phi_{er1}+\Phi_{er1}'\right)N_2\iota
\end{aligned}
$$

Note that (91) is the same for both countries. (62) then implies that aggregate expected excess returns are zero:

$$\left[\bar{E}_t^H er_{t+1}\right](2)=\left[\bar{E}_t^F er_{t+1}\right](2)=0$$

As (91) is zero, all its coefficients must be zero. $\alpha_{er}\left(0\right)=0$ was already derived in the first-order solution.

(90) can be split into the common components and the idiosyncratic ones, which only enter through $\left[E_t^{Hj}S_{t+1}\right](2)$. Using (79) and the results from the signal extraction, we get:

$$
\begin{aligned}
\left[E_t^{Hj}er_{t+1}\right](2) &= \left[\bar{E}_t^H er_{t+1}\right](2)+\left[\left(1-r_q\right)\left(I_1-\omega I_3\right)+r_q\alpha_{qD}\left(0\right)\right]N_2\left(0\right)\left[E_t^{Hj}\varepsilon_{t+1}\right](2) \\
&= \Theta_{z,er}\left(\frac{\epsilon_{j,t}^{H,H}}{\sigma_{H,H}^2}-\frac{\epsilon_{j,t}^{H,F}}{\sigma_{H,F}^2}\right) \qquad (93)
\end{aligned}
$$

where the second order coefficient $\Theta_{z,er}$ is:

$$\Theta_{z,er} = [1 - r_q + r_q \alpha_{1,qD}(0)] \frac{2\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2$$

Similarly we get:

$$\begin{aligned} \left[E_t^{Fj} er_{t+1}\right](2) &= [(1-r_q)(I_1 - \omega I_3) + r_q \alpha_{qD}(0)] N_2(0) \left[E_t^{Fj}\varepsilon_{t+1}\right](2) \\ &= \Theta_{z,er}\left(\frac{\epsilon_{j,t}^{F,H}}{\sigma_{H,F}^2} - \frac{\epsilon_{j,t}^{F,F}}{\sigma_{H,H}^2}\right) \end{aligned} \tag{94}$$

6.6 Dispersion of expected returns

The cross-sectional (fourth order) variance of expected average returns across Home investors is computed from (85):

$$\begin{aligned} Var_t^H\left(r_{t+1}^A\right) &= \int \left(\left[E_t^{Hj} r_{t+1}^A\right](2) - \left[\bar{E}_t r_{t+1}^A\right](2)\right)^2 dj \\ &= \left(\Theta_{z,ra}\right)^2 \frac{\sigma_{H,H}^2 + \sigma_{H,F}^2}{\sigma_{H,H}^2 \sigma_{H,F}^2} \end{aligned}$$

as $\sigma_{H,H}^2 = \int \left(\epsilon_{j,t}^{H,H}\right)^2 dj$ and $\sigma_{H,F}^2 = \int \left(\epsilon_{j,t}^{H,F}\right)^2 dj$. The cross-sectional variance of expected excess returns follows from (93):

$$\begin{aligned} Var_t^H\left(er_{t+1}\right) &= \int \left(\left[E_t^{Hj} er_{t+1}\right](2)\right)^2 dj \\ &= \left(\Theta_{z,er}\right)^2 \frac{\sigma_{H,H}^2 + \sigma_{H,F}^2}{\sigma_{H,H}^2 \sigma_{H,F}^2} \end{aligned}$$

The cross sectional variances across Foreign investors are identical.

As $r_{H,t+1} = r_{t+1}^A + 0.5er_{t+1}$ and $r_{F,t+1} = r_{t+1}^A - 0.5er_{t+1}$, the expected returns on Home and Foreign equity from the point of view of a Home investor are:

$$\begin{aligned} \left[E_t^{Hj} r_{H,t+1}\right](2) &= \left[\bar{E}_t r_{t+1}^A\right](2) + \frac{1}{2}\left[\bar{E}_t er_{t+1}\right](2) \\ &\quad + \left(\Theta_{z,ra} + \frac{\Theta_{z,er}}{2}\right)\frac{\epsilon_{j,t}^{H,H}}{\sigma_{H,H}^2} + \left(\Theta_{z,ra} - \frac{\Theta_{z,er}}{2}\right)\frac{\epsilon_{j,t}^{H,F}}{\sigma_{H,F}^2} \\ \left[E_t^{Hj} r_{F,t+1}\right](2) &= \left[\bar{E}_t r_{t+1}^A\right](2) - \frac{1}{2}\left[\bar{E}_t er_{t+1}\right](2) \\ &\quad + \left(\Theta_{z,ra} - \frac{\Theta_{z,er}}{2}\right)\frac{\epsilon_{j,t}^{H,H}}{\sigma_{H,H}^2} + \left(\Theta_{z,ra} + \frac{\Theta_{z,er}}{2}\right)\frac{\epsilon_{j,t}^{H,F}}{\sigma_{H,F}^2} \end{aligned}$$

The cross-sectional variances of these expected returns are computed as:

$$\begin{aligned}
Var_t^H\left(r_{H,t+1}\right) &= \left(\Theta_{z,ra}+\frac{\Theta_{z,er}}{2}\right)^2\frac{1}{\sigma_{H,H}^2}+\left(\Theta_{z,ra}-\frac{\Theta_{z,er}}{2}\right)^2\frac{1}{\sigma_{H,F}^2} \\
Var_t^H\left(r_{F,t+1}\right) &= \left(\Theta_{z,ra}-\frac{\Theta_{z,er}}{2}\right)^2\frac{1}{\sigma_{H,H}^2}+\left(\Theta_{z,ra}+\frac{\Theta_{z,er}}{2}\right)^2\frac{1}{\sigma_{H,F}^2}
\end{aligned}$$

Expectations are more dispersed for domestic than foreign returns, as investors put more weight on their signals on domestic productivity because they are more precise:

$$\begin{aligned}
Var_t^H\left(r_{H,t+1}\right)-Var_t^H\left(r_{F,t+1}\right) &= 2\Theta_{z,ra}\Theta_{z,er}\frac{\sigma_{H,F}^2-\sigma_{H,H}^2}{\sigma_{H,H}^2\sigma_{H,F}^2}>0 \\
Var_t^F\left(r_{H,t+1}\right)-Var_t^F\left(r_{F,t+1}\right) &= -\left[Var_t^H\left(r_{H,t+1}\right)-Var_t^H\left(r_{F,t+1}\right)\right]
\end{aligned}$$

6.7 Zero order portfolios

Using (60), (61) and the individual expected excess returns, the portfolio shares (42) and (43) are:

$$\begin{aligned}
z_{Hj}(0) &= \frac{1}{2}+\frac{\left[E_t^{Hj}er_{t+1}\right](2)+\tau}{\gamma\left[E_t\left(er_{t+1}\right)^2\right](2)} \\
&= \frac{1}{2}+\frac{\tau}{\gamma\left[E_t\left(er_{t+1}\right)^2\right](2)}+\frac{\Theta_{z,er}}{\gamma\left[E_t\left(er_{t+1}\right)^2\right](2)}\left(\frac{\epsilon_{j,t}^{H,H}}{\sigma_{H,H}^2}-\frac{\epsilon_{j,t}^{H,F}}{\sigma_{H,F}^2}\right) \quad (95) \\
z_{Fj}(0) &= \frac{1}{2}-\frac{\tau}{\gamma\left[E_t\left(er_{t+1}\right)^2\right](2)}+\frac{\Theta_{z,er}}{\gamma\left[E_t\left(er_{t+1}\right)^2\right](2)}\left(\frac{\epsilon_{j,t}^{F,H}}{\sigma_{H,F}^2}-\frac{\epsilon_{j,t}^{F,F}}{\sigma_{H,H}^2}\right) \quad (96)
\end{aligned}$$

Recalling that the private signals on Home and Foreign productivities are independent, the cross-sectional variances of portfolio shares are:

$$\begin{aligned}
\int\left(z_{Hj}(0)-z_H(0)\right)^2dj &= \left(\frac{\Theta_{z,er}}{\gamma\left[E_t\left(er_{t+1}\right)^2\right](2)}\right)^2\left[\int\left(\frac{\epsilon_{j,t}^{H,H}}{\sigma_{H,H}^2}\right)^2dj+\int\left(\frac{\epsilon_{j,t}^{H,F}}{\sigma_{H,F}^2}\right)^2dj\right] \\
&= \left(\frac{\Theta_{z,er}}{\gamma\left[E_t\left(er_{t+1}\right)^2\right](2)}\right)^2\frac{\sigma_{H,F}^2+\sigma_{H,H}^2}{\sigma_{H,H}^2\sigma_{H,F}^2}
\end{aligned}$$

The expression is the same in the Foreign country.

We now compute several expressions combining individual portfolio shares and expected excess returns that will be useful. Recalling that $z_H(0) = \int z_{Hj}(0)\,dj$ we write:

$$
\begin{aligned}
& \int \frac{2z_{Hj}(0)-1}{2}\left[E_t^{Hj} er_{t+1}\right](2)\,dj \\
= & \int \frac{2z_H(0)-1}{2}\left[E_t^{Hj} er_{t+1}\right](2)\,dj + \int \left(z_{Hj}(0) - z_H(0)\right)\left[E_t^{Hj} er_{t+1}\right](2)\,dj \\
= & \frac{2z_H(0)-1}{2}\left[\bar{E}_t^{H} er_{t+1}\right](2) + \frac{\left(\Theta_{zj}\right)^2}{\gamma\left[E_t\left(er_{t+1}\right)^2\right](2)} \int \left(\frac{\epsilon_{j,t}^{H,H}}{\sigma_{H,H}^2} - \frac{\epsilon_{j,t}^{H,F}}{\sigma_{H,F}^2}\right)^2 dj \\
= & \frac{\left(\Theta_{zj}\right)^2}{\gamma\left[E_t\left(er_{t+1}\right)^2\right](2)} \frac{\sigma_{H,F}^2 + \sigma_{H,H}^2}{\sigma_{H,H}^2 \sigma_{H,F}^2}
\end{aligned}
$$

where we used $\left[\bar{E}_t^{H} er_{t+1}\right](2) = 0$. Similarly:

$$
\int \frac{2z_{Fj}(0)-1}{2}\left[E_t^{Fj} er_{t+1}\right](2)\,dj = \frac{\left(\Theta_{zj}\right)^2}{\gamma\left[E_t\left(er_{t+1}\right)^2\right](2)} \frac{\sigma_{H,F}^2 + \sigma_{H,H}^2}{\sigma_{H,H}^2 \sigma_{H,F}^2}
$$

Next, we compute:

$$
\begin{aligned}
& \int z_{Hj}(0)\left(1 - z_{Hj}(0)\right) dj \\
= & \; z_H(0)\left(1 - z_H(0)\right) - \int \left(z_{Hj}(0) - z_H(0)\right)^2 dj \\
= & \; z_H(0)\left(1 - z_H(0)\right) - \left(\frac{\Theta_{z,er}}{\gamma\left[E_t\left(er_{t+1}\right)^2\right](2)}\right)^2 \frac{\sigma_{H,F}^2 + \sigma_{H,H}^2}{\sigma_{H,H}^2 \sigma_{H,F}^2}
\end{aligned}
$$

Similarly:

$$
\int z_{Fj}(0)\left(1 - z_{Fj}(0)\right) dj = z_F(0)\left(1 - z_F(0)\right) - \left(\frac{\Theta_{z,er}}{\gamma\left[E_t\left(er_{t+1}\right)^2\right](2)}\right)^2 \frac{\sigma_{H,F}^2 + \sigma_{H,H}^2}{\sigma_{H,H}^2 \sigma_{H,F}^2}
$$

6.8 Average consumption

We start from the second order component of (37):

$$
\begin{aligned}
& c_{yt}^{A}(2) \\
= \; & [I_2 + (1-\omega) I_4] \, S_t(2) \\
& + \frac{\gamma - \bar{c}}{2(1-\bar{c})} S_t'(1) \begin{bmatrix} (1-\alpha_{2,cA}(0)) I_2 \\ +(1-\omega-\alpha_{4,cA}(0)) I_4 \end{bmatrix}' \begin{bmatrix} (1-\alpha_{2,cA}(0)) I_2 \\ +(1-\omega-\alpha_{4,cA}(0)) I_4 \end{bmatrix} S_t(1) \\
& + \frac{1-\gamma}{\gamma} \frac{1-\bar{c}}{2} \left(1 - z^{D}(0)\right) \tau(2) \\
& - \frac{1-\gamma}{\gamma} \frac{1-\bar{c}}{2} \begin{bmatrix} \int \left[E_t^{Hj} r_{t+1}^{A}\right](2)\, dj + \int \frac{2z_{Hj}(0)-1}{2} \left[E_t^{Hj} er_{t+1}\right](2)\, dj \\ + \int \frac{z_{Hj}(0)\left(1-z_{Hj}(0)\right)}{2} E_t^{Hj}\left[(er_{t+1})^2\right](2)\, dj + \int E_t^{Hj}\left[z_{Hj,t} er_{t+1}\right](2)\, dj \\ + \int \left[E_t^{Fj} r_{t+1}^{A}\right](2)\, dj + \int \frac{2z_{Fj}(0)-1}{2} \left[E_t^{Fj} er_{t+1}\right](2)\, dj \\ + \int \frac{z_{Fj}(0)\left(1-z_{Fj}(0)\right)}{2} E_t^{Fj}\left[(er_{t+1})^2\right](2)\, dj + \int E_t^{Hj}\left[z_{Fj,t} er_{t+1}\right](2)\, dj \end{bmatrix} \\
& - \frac{1-\bar{c}}{4} \frac{(1-\gamma)^2}{\gamma} \begin{bmatrix} \int E_t^{Hj} \left[r_{t+1}^{A} + \frac{2z_{Hj}(0)-1}{2} er_{t+1}\right]^2 (2)\, dj \\ + \int E_t^{Fj} \left[r_{t+1}^{A} + \frac{2z_{Fj}(0)-1}{2} er_{t+1}\right]^2 (2)\, dj \end{bmatrix}
\end{aligned}
$$

We now take a closer look at the expectation terms. First, we write (recall that $z_{Hj,t}$ is in terms of deviations from the steady state):

$$
\left[E_t^{Hj} z_{Hj,t} er_{t+1}\right](2) = z_{Hj,t}(1) \left[E_t^{Hj} er_{t+1}\right](1) = 0
$$

Next, we use (61) and (84) to write:

$$
\begin{aligned}
& \int E_t^{Hj} \left[r_{t+1}^{A} + \frac{2z_{Hj}(0)-1}{2} er_{t+1}\right]^2 (2)\, dj \\
= \; & \int E_t^{Hj} \left[r_{t+1}^{A}\right]^2 (2)\, dj + E_t\left[er_{t+1}\right]^2 (2) \int \left(\frac{2z_{Hj}(0)-1}{2}\right)^2 dj \\
= \; & \left[E_t \left(r_{t+1}^{A}\right)^2\right](2) + E_t\left[er_{t+1}\right]^2 (2) \int \frac{1 - 4z_{Hj}(0)\left(1-z_{Hj}(0)\right)}{4} dj \\
= \; & \left[E_t \left(r_{t+1}^{A}\right)^2\right](2) + E_t\left[er_{t+1}\right]^2 (2) \left[\frac{2z_{H}(0)-1}{2}\right]^2 \\
& + E_t\left[er_{t+1}\right]^2 (2) \left(\frac{\Theta_{z,er}}{\gamma\left[E_t\left(er_{t+1}\right)^2\right](2)}\right)^2 \frac{\sigma_{H,F}^2 + \sigma_{H,H}^2}{\sigma_{H,H}^2 \sigma_{H,F}^2}
\end{aligned}
$$

Similarly:

$$\begin{aligned}
& \int E_t^{Fj}\left[r_{t+1}^A+\frac{2z_{Fj}(0)-1}{2}er_{t+1}\right]^2(2)\,dj \\
= & \left[E_t\left(r_{t+1}^A\right)^2\right](2)+E_t\left[er_{t+1}\right]^2(2)\left[\frac{2z_F(0)-1}{2}\right]^2 \\
& +E_t\left[er_{t+1}\right]^2(2)\left(\frac{\Theta_{z,er}}{\gamma\left[E_t\left(er_{t+1}\right)^2\right](2)}\right)^2\frac{\sigma_{H,F}^2+\sigma_{H,H}^2}{\sigma_{H,H}^2\sigma_{H,F}^2}
\end{aligned}$$

Putting our results together, we write the second order component of (37) as:

$$\begin{aligned}
& c_{yt}^A(2) \\
= & \left[I_2+(1-\omega)I_4\right]S_t(2) \\
& +\frac{\gamma-\bar{c}}{2(1-\bar{c})}S_t'(1)\begin{bmatrix}(1-\alpha_{2,cA}(0))I_2\\+(1-\omega-\alpha_{4,cA}(0))I_4\end{bmatrix}'\begin{bmatrix}(1-\alpha_{2,cA}(0))I_2\\+(1-\omega-\alpha_{4,cA}(0))I_4\end{bmatrix}S_t(1) \\
& -\frac{1-\gamma}{\gamma}(1-\bar{c})\left[\bar{E}_t r_{t+1}^A\right](2)-\frac{1-\bar{c}}{2}\frac{(1-\gamma)^2}{\gamma}\left[E_t\left(r_{t+1}^A\right)^2\right](2)+c(2)
\end{aligned}$$

where:

$$c(2)=\frac{1-\gamma}{\gamma}\frac{1-\bar{c}}{2}\begin{bmatrix}\left(1-z^D(0)\right)\tau(2)-E_t\left[er_{t+1}\right]^2(2)\frac{1-\gamma\left[z^D(0)\right]^2}{4}\\-E_t\left[er_{t+1}\right]^2(2)\,\gamma\left(\frac{\Theta_{z,er}}{\gamma\left[E_t(er_{t+1})^2\right](2)}\right)^2\frac{\sigma_{H,F}^2+\sigma_{H,H}^2}{\sigma_{H,H}^2\sigma_{H,F}^2}\end{bmatrix}$$

Recall that the conjecture for consumption is:

$$\begin{aligned}
c_{yt}^A(2) = & \alpha_{cA}(0)S_t(2)+\alpha_{cA}(1)S_t(1)+\alpha_{5cA}(1)x_t^D(1)+S_t'(1)A_{cA}(0)S_t(1) \\
& +\beta_{cA}(0)S_t(1)x_t^D(1)+\mu_{cA}(0)\left(x_t^D(1)\right)^2+\kappa_{cA}(2)
\end{aligned}$$

Using (84) and (87), as well as (73) and (79) and (81) and equalizing coefficients, we get:

$$\begin{aligned}
\alpha_{cA}(0) &= I_2+(1-\omega)I_4 \\
&+\frac{1-\gamma}{\gamma}(1-\bar{c})\left[\alpha_{qA}(0)-\left[(1-r_q)(I_2-\omega I_4)+r_q\alpha_{qA}(0)\right]N_1(0)\right] \\
\alpha_{cA}(1) &= \frac{1-\gamma}{\gamma}(1-\bar{c})\left[\begin{array}{c}\alpha_{qA}(1)-\left[(1-r_q)(I_2-\omega I_4)+r_q\alpha_{qA}(0)\right]N_1(1)\\ -r_q\alpha_{qA}(1)N_1(0)\end{array}\right] \\
\alpha_{5cA}(1) &= \frac{1-\gamma}{\gamma}(1-\bar{c})\left[\begin{array}{c}\alpha_{5,qA}(1)-\left[(1-r_q)(I_2-\omega I_4)+r_q\alpha_{qA}(0)\right]N_3(1)\\ -r_q\alpha_{qA}(1)\left[N_2\frac{1}{2\left(1+2\lambda^2\theta\right)}\iota+N_3(0)\right]x_t^D(1)\end{array}\right] \\
\kappa_{cA}(2)\frac{\gamma}{1-\gamma}\frac{1}{1-\bar{c}} &= (1-r_q)\kappa_{qA}-\left[(1-r_q)(I_2-\omega I_4)+r_q\alpha_{qA}(0)\right]\kappa(2) \\
&-\left(\left(\tilde{\Phi}_{ra1}\right)_{11}+\left(\tilde{\Phi}_{ra1}\right)_{22}\right)\frac{1+4\lambda^2\theta}{2\left(1+2\lambda^2\theta\right)}\sigma_a^2 \\
&-\left(\left(\tilde{\Phi}_{ra1}\right)_{12}+\left(\tilde{\Phi}_{ra1}\right)_{21}\right)\frac{1}{2\left(1+2\lambda^2\theta\right)}\sigma_a^2 \\
&-\left[r_q\mu_{qA}(0)+\frac{r_q(1-r_q)}{8}\left(\alpha_{5,qD}(0)\right)^2\right]2\sigma_a^2\left(1+2\lambda^2\theta\right) \\
&-\frac{1-\gamma}{2}\left[\begin{array}{c}\left(\left(\tilde{\Phi}_{ra2}\right)_{11}+\left(\tilde{\Phi}_{ra2}\right)_{22}\right)\frac{1+4\lambda^2\theta}{2\left(1+2\lambda^2\theta\right)}\sigma_a^2\\ +\left(\left(\tilde{\Phi}_{ra2}\right)_{12}+\left(\tilde{\Phi}_{ra2}\right)_{21}\right)\frac{1}{2\left(1+2\lambda^2\theta\right)}\sigma_a^2\end{array}\right] \\
&+\frac{1}{1-\bar{c}}\frac{\gamma}{1-\gamma}c(2)
\end{aligned} \tag{97}$$

In addition, we get:

$$\begin{aligned}
\mu_{cA}(0)\frac{\gamma}{1-\gamma}\frac{1}{1-\bar{c}} &= \mu_{qA}(0)-\left[(1-r_q)(I_2-\omega I_4)+r_q\alpha_{qA}(0)\right]N_4(0) \\
&-N_3(0)'\left[\Phi_{ra1}+\frac{1-\gamma}{2}\Phi_{ra2}\right]N_3(0) \\
&-\frac{1}{2\left(1+2\lambda^2\theta\right)}N_3(0)'\left[\Phi_{ra1}+\Phi_{ra1}'+\frac{1-\gamma}{2}\left(\Phi_{ra2}+\Phi_{ra2}'\right)\right]N_2\iota \\
&-\frac{\left(\tilde{\Phi}_{ra1}\right)_{11}+\left(\tilde{\Phi}_{ra1}\right)_{22}+\frac{1-\gamma}{2}\left[\left(\tilde{\Phi}_{ra2}\right)_{11}+\left(\tilde{\Phi}_{ra2}\right)_{22}\right]}{4\left(1+2\lambda^2\theta\right)^2} \\
&+\frac{\left(\tilde{\Phi}_{ra1}\right)_{12}+\left(\tilde{\Phi}_{ra1}\right)_{21}+\frac{1-\gamma}{2}\left[\left(\tilde{\Phi}_{ra2}\right)_{12}+\left(\tilde{\Phi}_{ra2}\right)_{21}\right]}{4\left(1+2\lambda^2\theta\right)^2}
\end{aligned}$$

as well as:

$$\begin{aligned}
&\beta_{cA}(0)\frac{\gamma}{1-\gamma}\frac{1}{1-\bar{c}} \\
= \; &\beta_{qA}(0) - \left[(1-r_q)(I_2-\omega I_4) + r_q\alpha_{qA}(0)\right] N_5(0) \\
&-\left[N_3(0) + \frac{1}{2\left(1+2\lambda^2\theta\right)} N_2\iota\right]' \left(\Phi_{ra1} + \Phi_{ra1}'\right) N_1(0) \\
&-\frac{1-\gamma}{2}\left[N_2\frac{1}{2\left(1+2\lambda^2\theta\right)}\iota + N_3(0)\right]' \left[\begin{array}{c} \left(\Phi_{ra2}+\Phi_{ra2}'\right) N_1(0) \\ -2\left[(1-r_q)(I_2-\omega I_4) + r_q\alpha_{qA}(0)\right]' \alpha_{qA}(0) \end{array}\right]
\end{aligned}$$

and finally:

$$\begin{aligned}
&A_{cA}(0)\frac{\gamma}{1-\gamma}\frac{1}{1-\bar{c}} \\
= \; &A_{qA}(0) + \frac{\gamma}{1-\gamma}\frac{\gamma-\bar{c}}{2(1-\bar{c})^2}\left[\begin{array}{c} (1-\alpha_{2,cA}(0)) I_2 \\ +(1-\omega-\alpha_{4,cA}(0))I_4 \end{array}\right]' \left[\begin{array}{c} (1-\alpha_{2,cA}(0)) I_2 \\ +(1-\omega-\alpha_{4,cA}(0))I_4 \end{array}\right] \\
&+\left[\omega(1-r_q) - r_q\alpha_{4,qA}(0)\right] A_{kA}(0) - N_1(0)' \left[\Phi_{ra1} + \frac{1-\gamma}{2}\Phi_{ra2}\right] N_1(0) \\
&-\frac{1-\gamma}{2}\alpha_{qA}'(0)\,\alpha_{qA}(0) + (1-\gamma)\,\alpha_{qA}'(0)\left[(1-r_q)(I_2-\omega I_4) + r_q\alpha_{qA}(0)\right] N_1(0)
\end{aligned}$$

6.9 Equalizing coefficients

6.9.1 $\alpha(1)$'s coefficients

From (76), (77), (78) we have series of coefficients on q^A, k^A and k^D. We also get the coefficients in (92) which are all zero, and the coefficients in (97).

For the various variables s, the $\alpha_s(0)$ coefficients have already been computed in the first-order solution. We next asset the $\alpha_s(1)$ coefficients, with the following system:

$$\begin{aligned}
\alpha_{qA}(1) &= -\frac{\xi}{1+\xi}\frac{\bar{c}}{1-\bar{c}}\alpha_{cA}(1) \\
\alpha_{kA}(1) &= \frac{1}{\xi}\alpha_{qA}(1) \\
\alpha_{kD}(1) &= \frac{1}{\xi}\alpha_{qD}(1) \\
0 &= \left[\alpha_{qD}(1) - r_q\alpha_{qD}(1) N_1(0)\right] + \omega(1-r_q)\,\alpha_{kD}(1) - r_q\alpha_{3,qD}(0)\,\alpha_{kD}(1) \\
\alpha_{cA}(1) &= \frac{1-\gamma}{\gamma}(1-\bar{c})\left[\begin{array}{c} \alpha_{qA}(1) - r_q\alpha_{qA}(1) N_1(0) \\ -\left[r_q\alpha_{4,qA}(0) - \omega(1-r_q)\right]\alpha_{kA}(1) \end{array}\right]
\end{aligned}$$

where we used:

$$N_1(1) = \begin{vmatrix} 0 \\ 0 \\ \alpha_{kD}(1) \\ \alpha_{kA}(1) \end{vmatrix}$$

Focus on the coefficients on cross-country variables. We get:

$$0 = [\alpha_{qD}(1) - r_q \alpha_{qD}(1) N_1(0)] + \omega(1-r_q)\frac{1}{\xi}\alpha_{qD}(1) - r_q \alpha_{3,qD}(0)\frac{1}{\xi}\alpha_{qD}(1)$$

Using our results for $N_1(0)$, this gives a system of four equations:

$$\begin{aligned}
0 &= \left[1 - r_q\rho + \omega(1-r_q)\frac{1}{\xi} - r_q\alpha_{3,qD}(0)\frac{1}{\xi}\right]\alpha_{1,qD}(1) - r_q\alpha_{1,kD}(0)\alpha_{3,qD}(1) \\
0 &= \left[1 - r_q\rho + \omega(1-r_q)\frac{1}{\xi} - r_q\alpha_{3,qD}(0)\frac{1}{\xi}\right]\alpha_{2,qD}(1) - r_q\alpha_{2,kA}(0)\alpha_{4,qD}(1) \\
0 &= \left[1 - r_q\alpha_{3,kD}(0) + \omega(1-r_q)\frac{1}{\xi} - r_q\alpha_{3,qD}(0)\frac{1}{\xi}\right]\alpha_{3,qD}(1) \\
0 &= \left[1 - r_q\alpha_{4,kA}(0) + \omega(1-r_q)\frac{1}{\xi} - r_q\alpha_{3,qD}(0)\frac{1}{\xi}\right]\alpha_{4,qD}(1)
\end{aligned}$$

As $\alpha_{3,qD}(0) < 0$ and $\alpha_{3,kD}(0) \in (0,1)$ and $\alpha_{4,kA}(0) \in (0,1)$, the last two relations imply $\alpha_{3,qD}(1) = \alpha_{4,qD}(1) = 0$, with the first two implying that $\alpha_{1,qD}(1) = \alpha_{2,qD}(1) = 0$. Focusing to the coefficients on worldwide averages, we get:

$$\begin{aligned}
0 = &\left[1 + \frac{\xi}{1+\xi}\frac{\bar{c}}{1-\bar{c}}\frac{1-\gamma}{\gamma}(1-\bar{c})\left[1 - r_q\alpha_{4,qA}(0) + \omega(1-r_q)\right]\frac{1}{\xi}\right]\alpha_{qA}(1) \\
&-\frac{\xi}{1+\xi}\frac{\bar{c}}{1-\bar{c}}\frac{1-\gamma}{\gamma}(1-\bar{c})\, r_q\alpha_{qA}(1) N_1(0)
\end{aligned}$$

This again gives a system of four equations:

$$\begin{aligned}
0 = &\left[1 + \frac{\xi}{1+\xi}\frac{\bar{c}}{1-\bar{c}}\frac{1-\gamma}{\gamma}(1-\bar{c})\left[1 - r_q\rho - r_q\alpha_{4,qA}(0) + \omega(1-r_q)\right]\frac{1}{\xi}\right]\alpha_{1,qA}(1) \\
&-\frac{\xi}{1+\xi}\frac{\bar{c}}{1-\bar{c}}\frac{1-\gamma}{\gamma}(1-\bar{c})\, r_q\alpha_{1,kD}(0)\alpha_{3,qA}(1) \\
0 = &\left[1 + \frac{\xi}{1+\xi}\frac{\bar{c}}{1-\bar{c}}\frac{1-\gamma}{\gamma}(1-\bar{c})\left[1 - r_q\rho - r_q\alpha_{4,qA}(0) + \omega(1-r_q)\right]\frac{1}{\xi}\right]\alpha_{2,qA}(1) \\
&-\frac{\xi}{1+\xi}\frac{\bar{c}}{1-\bar{c}}\frac{1-\gamma}{\gamma}(1-\bar{c})\, r_q\alpha_{2,kA}(0)\alpha_{4,qA}(1) \\
0 = &\left[1 + \frac{\xi}{1+\xi}\frac{\bar{c}}{1-\bar{c}}\frac{1-\gamma}{\gamma}(1-\bar{c})\left[1 - r_q\alpha_{3,kD}(0) - r_q\alpha_{4,qA}(0) + \omega(1-r_q)\right]\frac{1}{\xi}\right]\alpha_{3,qA}(1) \\
0 = &\left[1 + \frac{\xi}{1+\xi}\frac{\bar{c}}{1-\bar{c}}\frac{1-\gamma}{\gamma}(1-\bar{c})\left[1 - r_q\alpha_{4,kA}(0) - r_q\alpha_{4,qA}(0) + \omega(1-r_q)\right]\frac{1}{\xi}\right]\alpha_{4,qA}(1)
\end{aligned}$$

As $\alpha_{4,qA}(0) < 0$ and $\alpha_{3,kD}(0) \in (0,1)$ and $\alpha_{4,kA}(0) \in (0,1)$, the last two relations imply $\alpha_{3,qA}(1) = \alpha_{4,qA}(1) = 0$, with the first two implying that $\alpha_{1,qA}(1) = \alpha_{2,qA}(1) = 0$. We therefore get:

$$\alpha_{qD}(1) = \alpha_{kD}(1) = \alpha_{qA}(1) = \alpha_{kA}(1) = \alpha_{cA}(1) = 0 \tag{98}$$

6.9.2 $\alpha_5(1)$'s coefficients

We now turn to the $\alpha_{5,s}(1)$ coefficients, with the following system (using (98), which implies that $N_1(1) = 0$):

$$\begin{aligned}
\alpha_{5,qA}(1) &= -\frac{\xi}{1+\xi}\frac{\bar{c}}{1-\bar{c}}\alpha_{5,cA}(1) \\
\alpha_{5,kA}(1) &= \frac{1}{\xi}\alpha_{5,qA}(1) \\
\alpha_{5,kD}(1) &= \frac{1}{\xi}\alpha_{5,qD}(1) \\
0 &= \alpha_{5,qD}(1) + \omega(1-r_q)\alpha_{5,kD}(1) - r_q\alpha_{qD}(0)N_3(1) \\
\alpha_{cA}(1) &= 0
\end{aligned}$$

The worldwide averages are easy. Turning to the cross-country differences, we get:

$$\begin{aligned}
0 &= \alpha_{5,qD}(1) + \omega(1-r_q)\frac{1}{\xi}\alpha_{5,qD}(1) - r_q\alpha_{qD}(0)N_3(1) \\
0 &= \left[1 + \omega(1-r_q)\frac{1}{\xi} - r_q\alpha_{3,qD}(0)\frac{1}{\xi}\right]\alpha_{5,qD}(1)
\end{aligned}$$

As $\alpha_{3,qD}(0) < 0$ we immediately get $\alpha_{5,qD}(1) = \alpha_{5,kD}(1) = 0$. We therefore get:

$$\alpha_{5,qD}(1) = \alpha_{5,kD}(1) = \alpha_{5,qA}(1) = \alpha_{5,kA}(1) = \alpha_{5,cA}(1) = 0 \tag{99}$$

6.9.3 $A(0)$'s coefficients

We now turn to the matrices $A_s(0)$. It is useful to split the matrices between terms that we already know, denoted by "*other*" and terms that we still have to solve for. From (76), (77) and (78) we get:

$$\begin{aligned}
A_{qA}(0) &= -\frac{\xi}{1+\xi}\frac{\bar{c}}{1-\bar{c}}A_{cA}(0) + A_{qA}(other) \\
A_{kA}(0) &= \frac{1}{\xi}A_{qA}(0) + A_{kA}(other) \\
A_{kD}(0) &= \frac{1}{\xi}A_{qD}(0) + A_{kD}(other)
\end{aligned}$$

Using (89) and (92) we write:

$$0 = A_{qD}(0) - r_q N_1(0)' A_{qD}(0) N_1(0) + \omega(1-r_q) A_{kD}(0) - r_q \alpha_{3,qD}(0) A_{kD}(0) - A_{er}(other)$$

where:

$$A_{er}(other) = r_q(1-r_q) N_1(0)' [\alpha_{qA} - (I_2 - \omega I_4)]' [\alpha_{qD} - (I_1 - \omega I_3)] N_1(0) \tag{100}$$

Using (82) and (97) we write:

$$\begin{aligned} A_{cA}(0) \frac{\gamma}{1-\gamma} \frac{1}{1-\bar{c}} &= A_{qA}(0) + [\omega(1-r_q) - r_q \alpha_{4,qA}(0)] A_{kA}(0) \\ &\quad - r_q N_1(0)' A_{qA}(0) N_1(0) + A_{cA}(other) \end{aligned}$$

We start with the worldwide average coefficients, and write:

$$\begin{aligned} A_{cA}(0) \frac{\gamma}{1-\gamma} \frac{1}{1-\bar{c}} &= -\frac{\xi}{1+\xi} \frac{\bar{c}}{1-\bar{c}} A_{cA}(0) \\ &\quad - [\omega(1-r_q) - r_q \alpha_{4,qA}(0)] \frac{1}{1+\xi} \frac{\bar{c}}{1-\bar{c}} A_{cA}(0) \\ &\quad + r_q \frac{\xi}{1+\xi} \frac{\bar{c}}{1-\bar{c}} N_1(0)' A_{cA}(0) N_1(0) + A_{cA2}(other) \end{aligned} \tag{101}$$

where:

$$\begin{aligned} A_{cA2}(other) &= A_{qA}(other) + A_{cA}(other) - r_q N_1(0)' A_{qA}(other) N_1(0) \\ &\quad + [\omega(1-r_q) - r_q \alpha_{4,qA}(0)] \left[A_{kA}(other) + \frac{1}{\xi} A_{qA}(other) \right] \end{aligned}$$

The relation (101) is of the form:

$$\chi_A A_{cA}(0) = A_{cA3}(other) + N_1(0)' A_{cA}(0) N_1(0)$$

where χ_A is a scalar and:

$$\begin{aligned} \chi_A &= \frac{\frac{\gamma}{1-\gamma} \frac{1}{1-\bar{c}} + \frac{\xi}{1+\xi} \frac{\bar{c}}{1-\bar{c}} + [\omega(1-r_q) - r_q \alpha_{4,qA}(0)] \frac{1}{1+\xi} \frac{\bar{c}}{1-\bar{c}}}{r_q \frac{\xi}{1+\xi} \frac{\bar{c}}{1-\bar{c}}} \\ A_{cA3}(other) &= \frac{A_{cA2}(other)}{r_q \frac{\xi}{1+\xi} \frac{\bar{c}}{1-\bar{c}}} \end{aligned}$$

(101) is the implicit solution for the matrix $A_{cA}(0)$. We solve it using a vectorization method. Recall that if $n_{1,i}$'s are $4x1$ vectors and $A_{cA}(0)$ is a $4x4$ matrix:

$$(n_{1,i})' A_{cA}(0) (n_{1,j}) = \left[(n_{1,i})' A_{cA}(0) (n_{1,j}) \right]^{vec} = ([A_{cA}(0)]^{vec})' \left[(n_{1,i}) (n_{1,j})' \right]^{vec}$$

Now consider a $4x4$ matrix N_1 of which n_i is the ith column. $N_1' A_{cA}(0) N_1$ is then a $4x4$ matrix of which the ij cell (row i column j) is equal to $(n_{1,i})' A_{cA}(0)(n_{1,j})$. Hence the ij cell is equal to $([A_{cA}(0)]^{vec})' \left[(n_{1,i})(n_{1,j})'\right]^{vec}$. We then write:

$$\left[N_1(0)' A_{cA}(0) N_1(0)\right]^{vec} = \begin{vmatrix} ([A_{cA}(0)]^{vec})' \left[(n_{1,1})(n_{1,1})'\right]^{vec} \\ ([A_{cA}(0)]^{vec})' \left[(n_{1,2})(n_{1,1})'\right]^{vec} \\ ([A_{cA}(0)]^{vec})' \left[(n_{1,3})(n_{1,1})'\right]^{vec} \\ \ldots \end{vmatrix} = \begin{vmatrix} ([A_{cA}(0)]^{vec})' k_1 \\ ([A_{cA}(0)]^{vec})' k_2 \\ ([A_{cA}(0)]^{vec})' k_3 \\ \ldots \end{vmatrix}$$

where each k_i is a $16x1$ vector that we compute from $N_1(0)$. (101) is put in a vectorized form as:

$$\chi_A [A_{cA}(0)]^{vec} = [A_{cA3}(other)]^{vec} + \left[N_1(0)' A_{cA}(0) N_1(0)\right]^{vec}$$

This gives a system of 16 equations:

$$\begin{aligned} \chi_A [A_{cA}(0)]_1^{vec} &= [A_{cA3}(other)]_1^{vec} + ([A_{cA}(0)]^{vec})' k_1 \\ \chi_A [A_{cA}(0)]_2^{vec} &= [A_{cA3}(other)]_2^{vec} + ([A_{cA}(0)]^{vec})' k_2 \\ \chi_A [A_{cA}(0)]_3^{vec} &= [A_{cA3}(other)]_3^{vec} + ([A_{cA}(0)]^{vec})' k_3 \\ &\ldots. \end{aligned}$$

Denote by $\bar{I}_i$ a $16x1$ vector with zeros everywhere and 1 in the ith column. We then write: $[A_{cA}(0)]_i^{vec} = (\bar{I}_i)' [A_{cA}(0)]^{vec}$. Our 16 equations then become:

$$\left(\chi_A (\bar{I}_i)' - (k_i)'\right) [A_{cA}(0)]^{vec} = [A_{cA3}(other)]_i^{vec}$$

Stacking these equations in a matrix form we solve for $A_{cA}(0)$:

$$X_A [A_{cA}(0)]^{vec} = [A_{cA3}(other)]^{vec} \Longrightarrow [A_{cA}(0)]^{vec} = (X_A)^{-1} [A_{cA3}(other)]^{vec} \tag{102}$$

where X_A is a $16x16$ matrix where the i row is $x_A (\bar{I}_i)' - (k_i)'$. $A_{cA}(0)$ is simply obtained by "de-vectorizing" $[A_{cA}(0)]^{vec}$. $A_{qA}(0)$ and $A_{kA}(0)$ easily follow.

We next move to the cross-country differences:

$$\begin{aligned} 0 &= A_{qD}(0) - r_q N_1(0)' A_{qD}(0) N_1(0) + [\omega(1-r_q) - r_q \alpha_{3,qD}(0)] A_{kD}(0) - A_{er}(other) \\ &= A_{qD}(0) - r_q N_1(0)' A_{qD}(0) N_1(0) + [\omega(1-r_q) - r_q \alpha_{3,qD}(0)] \frac{1}{\xi} A_{qD}(0) \\ &\quad + [\omega(1-r_q) - r_q \alpha_{3,qD}(0)] A_{kD}(other) - A_{er}(other) \end{aligned}$$

which we re-write as:

$$\chi_D A_{qD}(0) = A_{qD}(other) + N_1(0)' A_{qD}(0) N_1(0) \tag{103}$$

where:

$$\begin{aligned}
\chi_D &= \frac{\xi + \omega(1-r_q) - r_q \alpha_{3,qD}(0)}{r_q \xi} \\
A_{qD}(other) &= \frac{A_{er}(other) - [\omega(1-r_q) - r_q \alpha_{3,qD}(0)] A_{kD}(other)}{r_q}
\end{aligned}$$

This gives an implicit solution for $A_{qD}(0)$ which we solve using the same vectorization method as above, yielding:

$$[A_{qD}(0)]^{vec} = (X_D)^{-1} [A_{qD}(other)]^{vec} \tag{104}$$

where X_D is a $16x16$ matrix where the i row is $\chi_D \left(\bar{I}_i\right)' - (k_i)'$. $A_{kD}(0)$ immediately follows.

6.9.4 $\beta(0)$'s coefficients

From (102)-(104) we can compute the values of Φ_{ra1} and Φ_{er1} from (82) and (89). It is again useful to split the vectors between terms that we already know, denoted by "*other*" and terms that we still have to solve for:

$$\begin{aligned}
\beta_{qA}(0) &= -\frac{\xi}{1+\xi}\frac{\bar{c}}{1-\bar{c}}\beta_{cA}(0) + \beta_{qA}(other) \\
\beta_{kA}(0) &= \frac{1}{\xi}\beta_{qA}(0) + \beta_{kA}(other) \\
\beta_{kD}(0) &= \frac{1}{\xi}\beta_{qD}(0) + \beta_{kD}(other)
\end{aligned}$$

Recalling that $N_5(0) = \begin{vmatrix} 0 & 0 & \beta_{kD}(0) & \beta_{kA}(0) \end{vmatrix}'$, we also write:

$$0 = -\beta_{qD}(0) - \omega(1-r_q)\beta_{kD}(0) + r_q \alpha_{3,qD}(0)\beta_{kD}(0) + \beta_{er}(other)$$

where:

$$\beta_{er}(other) = \left[N_3(0) + \frac{1}{2\left(1+2\lambda^2\theta\right)} N_{2\iota} \right]' \left(\Phi_{er1} + \Phi'_{er1}\right) N_1(0)$$

and:

$$\beta_{cA}(0)\frac{\gamma}{1-\gamma}\frac{1}{1-\bar{c}}=\beta_{qA}(0)-\left[r_q\alpha_{4,qA}(0)-\omega(1-r_q)\right]\beta_{kA}(0)+\beta_{cA}(other)$$

Starting with the coefficients for worldwide averages, we get:

$$\beta_{cA}(0)=\frac{1}{D_{cA}}\left[\begin{array}{c}\beta_{cA}(other)-\left[r_q\alpha_{4,qA}(0)-\omega(1-r_q)\right]\beta_{kA}(other)\\ +\left[1-\left[r_q\alpha_{4,qA}(0)-\omega(1-r_q)\right]\frac{1}{\xi}\right]\beta_{qA}(other)\end{array}\right] \quad (105)$$

where:

$$D_{cA}=\frac{\gamma}{1-\gamma}\frac{1}{1-\bar{c}}+\left[1-\left[r_q\alpha_{4,qA}(0)-\omega(1-r_q)\right]\frac{1}{\xi}\right]\frac{\xi}{1+\xi}\frac{\bar{c}}{1-\bar{c}}$$

with $\beta_{qA}(0)$ and $\beta_{kA}(0)$ following easily.

Turning to the cross-country differences, we get:

$$\beta_{qD}(0)=\frac{\left[r_q\alpha_{3,qD}(0)-\omega(1-r_q)\right]\beta_{kD}(other)+\beta_{er}(other)}{1-\left[r_q\alpha_{3,qD}(0)-\omega(1-r_q)\right]\frac{1}{\xi}} \quad (106)$$

with $\beta_{kD}(0)$ following easily.

6.9.5 $\mu(0)$'s coefficients

It is again useful to split the vectors between terms that we already know, denoted by "*other*" and terms that we still have to solve for. Recalling that $N_4(0)=\left|\begin{array}{cccc}0 & 0 & \mu_{kD}(0) & \mu_{kA}(0)\end{array}\right|'$ we write:

$$\begin{aligned}
\mu_{qA}(0) &= -\frac{\xi}{1+\xi}\frac{\bar{c}}{1-\bar{c}}\mu_{cA}(0)+\mu_{qA}(other)\\
\mu_{kA}(0) &= \frac{1}{\xi}\mu_{qA}(0)+\mu_{kA}(other)\\
\mu_{kD}(0) &= \frac{1}{\xi}\mu_{qD}(0)\\
0 &= -\mu_{qD}(0)-\omega(1-r_q)\mu_{kD}(0)+r_q\alpha_{3,qD}(0)\mu_{kD}(0)+\mu_{er}(other)\\
\mu_{cA}(0)\frac{\gamma}{1-\gamma}\frac{1}{1-\bar{c}} &= \mu_{qA}(0)-\left[r_q\alpha_{4,qA}(0)-\omega(1-r_q)\right]\mu_{kA}(0)+\mu_{cA}(other)
\end{aligned}$$

where:

$$\begin{aligned}
\mu_{er}(other) &= \frac{\left[\tilde{\Phi}_{er1}\right]_{11}+\left[\tilde{\Phi}_{er1}\right]_{22}-\left[\tilde{\Phi}_{er1}\right]_{12}-\left[\tilde{\Phi}_{er1}\right]_{21}}{4\left(1+2\lambda^2\theta\right)^2}\\
&\quad +N_3(0)'\Phi_{er1}N_3(0)+\frac{1}{2\left(1+2\lambda^2\theta\right)}N_3(0)'\left(\Phi_{er1}+\Phi_{er1}'\right)N_2\iota
\end{aligned}$$

Starting with the coefficients for worldwide averages, we get:

$$\mu_{cA}(0) = \frac{1}{D_{cA}} \left[\begin{array}{c} \mu_{cA}(other) - [r_q \alpha_{4,qA}(0) - \omega(1-r_q)] \mu_{kA}(other) \\ + \left[1 - [r_q \alpha_{4,qA}(0) - \omega(1-r_q)] \frac{1}{\xi}\right] \mu_{qA}(other) \end{array} \right] \tag{107}$$

with D_{cA} is as in (105), and $\beta_{qA}(0)$ and $\beta_{kA}(0)$ following easily.

Turning to the cross-country differences, we get:

$$\mu_{qD}(0) = \frac{\mu_{er}(other)}{1 - [r_q \alpha_{3,qD}(0) - \omega(1-r_q)] \frac{1}{\xi}} \tag{108}$$

with $\mu_{kD}(0)$ following easily.

7 Overall solution

7.1 Third-order expected excess returns

While we have now solved for all coefficients in the first- and second-order solution, this solution remains conditional on the value of λ in (50).

Closing the model will require a solution for the third-order expected excess returns. As shown below, we only need to solve for their linear component, which we take from (30):

$$er_{t+1} = (1-r_q)\left[a^D_{t+1} - \omega k^D_{t+1}\right] + r_q q^D_{t+1} - q^D_t$$

The expectation for a Home investor is:

$$\begin{aligned} & \left[E^{Hj}_t er_{t+1}\right](3) \\ = \; & (1-r_q)\left[\rho a^D_t + \left[E^{Hj}_t \varepsilon^D_{t+1}\right](3) - \omega k^D_{t+1}\right] - q^D_t \\ & + r_q \left[\alpha_{qD}\left[E^{Hj}_t S_{t+1}\right](3) + \left[E^{Hj}_t S'_{t+1} A_{qD} S_{t+1}\right](3) + \mu_{qD}\left[E^{Hj}_t \left(x^D_{t+1}\right)^2\right](3) + \kappa_{qD}\right] \end{aligned}$$

where we used () and the fact that S_{t+1} and x^D_{t+1} are independent. As detailed below, our focus is on the terms where ε^D_{t+1} enters linearly. It does not enter terms set at time t, neither x^D_{t+1}. ε^D_{t+1} also does not enter cross-product terms linearly, such as $S'_{t+1} A_{qD} S_{t+1}$. We therefore rewrite the third-order expected returns as:

$$\left[E^{Hj}_t er_{t+1}\right](3) = (1-r_q)\left[E^{Hj}_t \varepsilon^D_{t+1}\right](3) + r_q \alpha_{qD}\left[E^{Hj}_t S_{t+1}\right](3) + other$$

where *other* denotes variables where ε_{t+1}^{D} does not enter linearly. Using (57), we focus on the terms of $\left[E_t^{Hj} S_{t+1}\right](3)$ where ε_{t+1}^{D} enters linearly:

$$\begin{aligned}\left[E_t^{Hj} er_{t+1}\right](3) &= (1-r_q)\left[E_t^{Hj}\varepsilon_{t+1}^{D}\right](3) + r_q\alpha_{qD}(0)\,N_2\left[E_t^{Hj}\varepsilon_{t+1}\right](3) + other \\ &= \left[1-r_q+r_q\alpha_{1,qD}(0)\right]\left[\left[E_t^{Hj}\varepsilon_{H,t+1}\right](3) - \left[E_t^{Hj}\varepsilon_{F,t+1}\right](3)\right] + other\end{aligned}$$

Using the results of the signal extraction, we write:

$$\begin{aligned}&\left[E_t^{Hj} er_{t+1}\right](3) \\ =\; & -\left[1-r_q+r_q\alpha_{1,qD}(0)\right]\frac{\lambda^2\theta\sigma_a^2}{\left(1+2\lambda^2\theta\right)^2}\frac{\sigma_{H,H}^2+\sigma_{H,F}^2}{\sigma_{H,H}^2\sigma_{H,F}^2}x_t^D(1) \\ & +\left[1-r_q+r_q\alpha_{1,qD}(0)\right]\frac{2\lambda^2\theta\sigma_a^2}{1+\lambda^2\theta\left(1-\rho_\tau\right)}\left[\frac{\varepsilon_{H,t+1}}{\sigma_{H,H}^2}-\frac{\varepsilon_{F,t+1}}{\sigma_{H,F}^2}\right] \\ & +other\end{aligned} \tag{109}$$

Similarly, we derive:

$$\begin{aligned}&\left[E_t^{Fj} er_{t+1}\right](3) \\ =\; & -\left[1-r_q+r_q\alpha_{1,qD}(0)\right]\frac{\lambda^2\theta\sigma_a^2}{\left(1+2\lambda^2\theta\right)^2}\frac{\sigma_{H,H}^2+\sigma_{H,F}^2}{\sigma_{H,H}^2\sigma_{H,F}^2}x_t^D(1) \\ & +\left[1-r_q+r_q\alpha_{1,qD}(0)\right]\frac{2\lambda^2\theta\sigma_a^2}{1+\lambda^2\theta\left(1-\rho_\tau\right)}\left[\frac{\varepsilon_{H,t+1}}{\sigma_{H,F}^2}-\frac{\varepsilon_{F,t+1}}{\sigma_{H,H}^2}\right] \\ & +other\end{aligned} \tag{110}$$

7.2 Solution for λ

We now turn to the solution for λ in (50). We combine the cross-country difference in asset market clearing conditions (36) with the first-order average portfolio share (48). Recalling that to a first-order the consumption of young agents is the same for all agents in a given country, and using (68), the first-order component of (36) is written as:

$$4z_t^A(1) = q_t^D(1) + k_{t+1}^D(1) - z^D(0)\left[a_t^D(1) + (1-\omega)\,k_t^D(1)\right]$$

Combining this with (48) we write:

$$\begin{aligned}
&\frac{\gamma}{2}\left[q_t^D(1)+k_{t+1}^D(1)-z^D(0)\left[a_t^D(1)+(1-\omega)k_t^D(1)\right]\right]\left[E_t\left(er_{t+1}\right)^2\right](2)\\
=\;&\left[\bar{E}_t^H er_{t+1}\right](3)+\left[\bar{E}_t^F er_{t+1}\right](3)+\tau_t^D(3)\\
&+(1-\gamma)\left[\left[\bar{E}_t^H r_{t+1}^A er_{t+1}\right](3)+\left[\bar{E}_t^F r_{t+1}^A er_{t+1}\right](3)\right]\\
&-\gamma\left[\begin{array}{c}\int\frac{2z_{Hj}(0)-1}{2}\left[E_t^{Hj}\left(er_{t+1}\right)^2\right](3)\,dj\\+\int\frac{2z_{Fj}(0)-1}{2}\left[E_t^{Fj}\left(er_{t+1}\right)^2\right](3)\,dj\end{array}\right]\\
&+\frac{(1-\gamma)^2}{2}\left[\left[\bar{E}_t^H\left(r_{t+1}^A\right)^2 er_{t+1}\right](3)+\left[\bar{E}_t^F\left(r_{t+1}^A\right)^2 er_{t+1}\right](3)\right]\\
&-\gamma(1-\gamma)\left[\begin{array}{c}\int\frac{2z_{Hj}(0)-1}{2}\left[E_t^{Hj}r_{t+1}^A\left(er_{t+1}\right)^2\right](3)\,dj\\+\int\frac{2z_{Fj}(0)-1}{2}\left[E_t^{Fj}r_{t+1}^A\left(er_{t+1}\right)^2\right](3)\,dj\end{array}\right]
\end{aligned} \tag{111}$$

where we used the fact that $\left[E_t^{Hj}r_{t+1}^A\right](1)=\left[E_t^{Fj}r_{t+1}^A\right](1)$.

We can infer λ from (111). The key aspect is that $\tau_t^D(3)$ and ε_{t+1}^D can enter (111) only in the same way as they enter $x_t^D(1)$, otherwise agents could split $x_t^D(1)$ between its components. ε_{t+1}^D clearly does not enter the left-hand side of (111), so we focus on the right-hand side.

$\tau_t^D(3)$ clearly enters through the third-order expected excess returns, including the iceberg cost, and in no other place. ε_{t+1}^D cannot enter directly through terms that are expectations of cross-products (as in lines 3 and following). Such terms would only lead to variances of shocks, or the expectation of ε_{t+1}^D which is a function of $x_t^D(1)$.

ε_{t+1}^D can only enter directly through the first-order component of private signals (19)-(20). We know from the signal extraction problem that the coefficient on private signals are second-order. Therefore, the combination of these coefficients and ε_{t+1}^D is third-order. ε_{t+1}^D then only enter directly through linear third-order terms, that is the third-order expected excess returns. We can therefore focus on the following part of (111):

$$0=\left[\bar{E}_t^H er_{t+1}\right](3)+\left[\bar{E}_t^F er_{t+1}\right](3)+\tau_t^D(3)$$

We can focus on the linear component of third-order expected excess returns, as again ε_{t+1}^D only enter directly through the first-order component of private signals, which is multiplied by second order coefficients.

Focusing on the terms in (109) and (110) where productivity innovations enter linearly, we write:

$$\begin{aligned} 0 &= \left[\bar{E}_t^H er_{t+1}\right](3) + \left[\bar{E}_t^F er_{t+1}\right](3) + \tau_t^D(3) \\ 0 &= \left[1 - r_q + r_q \alpha_{1,qD}(0)\right] \frac{2\lambda^2\theta\sigma_a^2}{1+2\lambda^2\theta} \frac{\sigma_{H,H}^2 + \sigma_{H,F}^2}{\sigma_{H,H}^2\sigma_{H,F}^2} \varepsilon_{t+1}^D + \tau_t^D(3) \end{aligned}$$

As ε_{t+1}^D and $\tau_t^D(3)$ must enter in the same way as through (50), we replace $\tau_t^D(3)$ by $-\varepsilon_{t+1}^D \tau(2)/\lambda$:

$$0 = \left[1 - r_q + r_q \alpha_{1,qD}(0)\right] \frac{2\lambda^2\theta\sigma_a^2}{1+2\lambda^2\theta} \frac{\sigma_{H,H}^2 + \sigma_{H,F}^2}{\sigma_{H,H}^2\sigma_{H,F}^2} \varepsilon_{t+1}^D - \varepsilon_{t+1}^D \frac{\tau(2)}{\lambda}$$

from which we get an implicit solution for λ:

$$\left[1 - r_q + r_q \alpha_{1,qD}(0)\right] \frac{\sigma_{H,H}^2 + \sigma_{H,F}^2}{\sigma_{H,H}^2\sigma_{H,F}^2} \lambda = \frac{1+2\lambda^2\theta}{2\lambda^2\theta} \frac{\tau(2)}{\sigma_a^2} \tag{112}$$

8 First-order portfolio shares

8.1 Useful expectations

The solution for $z_t^D(1)$ follows from (49), requiring us to compute the expectation of several combinations of the average return and excess returns from (29)-(30). We already now $\left[E_t(er_{t+1})^2\right](2)$ from (60).

8.1.1 Third order expected excess return

The third-order expected excess returns enter (49) only in terms of differences, $\left[\bar{E}_t^H er_{t+1}\right](3) - \left[\bar{E}_t^F er_{t+1}\right](3)$. From the signal extraction we have shown that agents agree on first-order expectations. Disagreements on the second-order expectations reflect the idiosyncratic component of signal, that adds up to zero in each country. Different third-order expected excess returns then reflect the linear component of (30):

$$er_{t+1} = (1 - r_q)\left[I_1 - \omega I_3\right] S_{t+1} + r_q \left[\alpha_{qD} S_{t+1} + \alpha_{5,qD} x_{t+1}^D\right] - q_t^D$$

There is no disagreement on publicly observed variables, and on x_{t+1}^D as no agent has any information on it. We therefore focus on:

$$er_{t+1} = (1 - r_q)\varepsilon_{t+1}^D + r_q \alpha_{qD}\left[N_1 S_t + N_2 \varepsilon_{t+1} + N_3 x_t^D\right]$$

where we used the linear component of (57). There is no disagreement on S_t and x_t^D which are publicly observed, hence we focus on:

$$er_{t+1} = (1 - r_q)\, \varepsilon_{t+1}^D + r_q \alpha_{qD} \begin{vmatrix} \varepsilon_{t+1}^D \\ \varepsilon_{t+1}^A \\ 0 \\ 0 \end{vmatrix}$$

Taking the third order expectation for a Home agent, we get:

$$\begin{aligned} & \left[E_t^{Hj} er_{t+1}\right](3) \\ = \;& (1 - r_q)\left[E_t^{Hj} \varepsilon_{t+1}^D\right](3) + r_q \alpha_{qD}(0) \begin{vmatrix} \left[E_t^{Hj} \varepsilon_{t+1}^D\right](3) \\ \left[E_t^{Hj} \varepsilon_{t+1}^A\right](3) \\ 0 \\ 0 \end{vmatrix} \\ = \;& (1 - r_q + r_q \alpha_{1,qD}(0))\left[E_t^{Hj} \varepsilon_{t+1}^D\right](3) \end{aligned}$$

which we already computed in (109). Similarly the expectation for a Foreign agent is given by (110). We therefore write:

$$\begin{aligned} & \left[\bar{E}_t^H er_{t+1}\right](3) - \left[\bar{E}_t^F er_{t+1}\right](3) \\ = \;& 2\left[1 - r_q + r_q \alpha_{1,qD}(0)\right] \frac{2\lambda^2 \theta \sigma_a^2}{1 + 2\lambda^2 \theta} \frac{\sigma_{H,F}^2 - \sigma_{H,H}^2}{\sigma_{H,H}^2 \sigma_{H,F}^2} \varepsilon_{t+1}^A \end{aligned} \tag{113}$$

8.1.2 First order expected average return

The first-order expected average return enter (49) in terms of the sum across all agents: $\left[\bar{E}_t^H r_{t+1}^A\right](1) + \left[\bar{E}_t^F r_{t+1}^A\right](1)$. We evaluate it using the linear terms in (29):

$$r_{t+1}^A = (1 - r_q)\left[a_{t+1}^A - \omega k_{t+1}^A\right] + r_q q_{t+1}^A - q_t^A$$

All agents agree on their first-order expectations. From the signal extraction we know that for any agent $\left[E_t a_{t+1}^A\right](1) = \rho a_t^A$. Therefore:

$$\begin{aligned}
& \left[E_t r_{t+1}^A\right](1) \\
= \; & (1-r_q)\left[\left[E_t a_{t+1}^A\right](1) - \omega k_{t+1}^A(1)\right] \\
& + r_q\left[\alpha_{2,qA}(0)\left[E_t a_{t+1}^A\right](1) + \alpha_{4,qA}(0)\, k_{t+1}^A(1)\right] - q_t^A(1) \\
= \; & \left[\begin{array}{c} \left[1 - r_q + r_q\alpha_{2,qA}(0)\right]\rho - \alpha_{2,qA}(0) \\ + \left[r_q\alpha_{4,qA}(0) - \omega(1-r_q)\right]\alpha_{2,kA}(0) \end{array}\right] a_t^A(1) \\
& + \left[\left[r_q\alpha_{4,qA}(0) - \omega(1-r_q)\right]\alpha_{4,kA}(0) - \alpha_{4,qA}(0)\right] k_t^A(1) \\
= \; & \alpha_{2,erA}(0)\, a_t^A(1) + \alpha_{4,erA}(0)\, k_t^A(1)
\end{aligned}$$

8.1.3 Expected $r_{t+1}^A(er_{t+1})^2$

We next turn to $\left[E_t^{Hj} r_{t+1}^A (er_{t+1})^2\right](3)$ and $\left[E_t^{Fj} r_{t+1}^A (er_{t+1})^2\right](3)$. These entail cubic product, and can then be evaluated using only the linear components of (29) and (30). Specifically, we use the fact that first-order expectations of the average are the same for all agents to write:

$$\begin{aligned}
r_{t+1}^A &= (1-r_q)\left[a_{t+1}^A - \omega k_{t+1}^A\right] + r_q q_{t+1}^A - q_t^A \\
&= \left[E_t r_{t+1}^A\right](1) + \left[1 - r_q + r_q\alpha_{2,qA}(0)\right]\varepsilon_{t+1}^A
\end{aligned}$$

Recall that for all agents first-order expected excess returns are zero, hence using (58):

$$\begin{aligned}
0 &= \left[E_t er_{t+1}\right](1) \\
&= \frac{1 - r_q + r_q\alpha_{1,qD}(0)}{1 + 2\lambda^2\theta} x_t^D(1) \\
&\quad + \left[1 - r_q + r_q\alpha_{1,qD}(0)\right]\rho a_t^D(1) + \left[r_q\alpha_{3,qD}(0) - \omega(1-r_q)\right] k_{t+1}^D(1) - q_t^D(1)
\end{aligned}$$

The actual excess returns are then:

$$\begin{aligned}
& er_{t+1} \\
= \; & (1-r_q)\left[a_{t+1}^D - \omega k_{t+1}^D\right] + r_q q_{t+1}^D - q_t^D \\
= \; & \left[1 - r_q + r_q\alpha_{1,qD}(0)\right]\varepsilon_{t+1}^D + r_q\alpha_{5,qD}(0)\, x_{t+1}^D(1) \\
& + \left[1 - r_q + r_q\alpha_{1,qD}(0)\right]\rho a_t^D(1) + \left[r_q\alpha_{3,qD}(0) - \omega(1-r_q)\right] k_{t+1}^D(1) - q_t^D(1) \\
= \; & \left[1 - r_q + r_q\alpha_{1,qD}(0)\right]\varepsilon_{t+1}^D + r_q\alpha_{5,qD}(0)\, x_{t+1}^D(1) - \frac{1 - r_q + r_q\alpha_{1,qD}(0)}{1 + 2\lambda^2\theta} x_t^D(1)
\end{aligned}$$

We can the write:

$$
\begin{aligned}
& r^A_{t+1}\left(er_{t+1}\right)^2 \\
= & \left[E_t r^A_{t+1}\right](1)\left(er_{t+1}\right)^2+\left[1-r_q+r_q\alpha_{2,qA}(0)\right]\varepsilon^A_{t+1}\left(er_{t+1}\right)^2 \\
= & \left[E_t r^A_{t+1}\right](1)\left(er_{t+1}\right)^2 \\
& +\left[1-r_q+r_q\alpha_{2,qA}(0)\right]\varepsilon^A_{t+1}\left[\begin{array}{c}
\left[\left[1-r_q+r_q\alpha_{1,qD}(0)\right]\varepsilon^D_{t+1}\right]^2+\left[\frac{1-r_q+r_q\alpha_{1,qD}(0)}{1+2\lambda^2\theta}x^D_t(1)\right]^2 \\
-2\left[1-r_q+r_q\alpha_{1,qD}(0)\right]\frac{1-r_q+r_q\alpha_{1,qD}(0)}{1+2\lambda^2\theta}\varepsilon^D_{t+1}x^D_t(1) \\
+\left[r_q\alpha_{5,qD}(0)\,x^D_{t+1}(1)\right]^2 \\
+2r_q\alpha_{5,qD}(0)\,x^D_{t+1}(1)\left[\begin{array}{c}\left[1-r_q+r_q\alpha_{1,qD}(0)\right]\varepsilon^D_{t+1}+ \\ -\frac{1-r_q+r_q\alpha_{1,qD}(0)}{1+2\lambda^2\theta}x^D_t(1)\end{array}\right]
\end{array}\right]
\end{aligned}
$$

We now take expectations from the point of view of a Home investor. We already now $\left[E_t\left(er_{t+1}\right)^2\right](2)$ from (60). We know that $\left[E_t\varepsilon^A_{t+1}\right](1)=0$. As $x^D_{t+1}(1)$ is independent from ε^A_{t+1}, we get $\left[E^{Hj}_t\varepsilon^A_{t+1}\left(x^D_{t+1}\right)^2\right](3)=\left[E_t\varepsilon^A_{t+1}\right](1)\left[E^{Hj}_t\left(x^D_{t+1}\right)^2\right](2)=0$, and similarly $\left[E^{Hj}_t\varepsilon^A_{t+1}x^D_{t+1}x^D_t\right](3)=x^D_t(1)\left[E^{Hj}_t\varepsilon^A_{t+1}\right](1)\left[E^{Hj}_t x^D_{t+1}\right](1)=0$. Using (58) we also get $\left[E^{Hj}_t\varepsilon^A_{t+1}\varepsilon^D_{t+1}x^D_{t+1}\right](3)=\left[E^{Hj}_t\varepsilon^A_{t+1}\varepsilon^D_{t+1}\right](2)\left[E^{Hj}_t x^D_{t+1}\right](1)=0$. From the signal extraction:

$$
\begin{aligned}
& \left[E^{Hj}_t\varepsilon^A_{t+1}\varepsilon^D_{t+1}x^D_t\right](3) \\
= & \; x^D_t(1)\left[E^{Hj}_t\varepsilon^A_{t+1}\varepsilon^D_{t+1}\right](2) \\
= & \; x^D_t(1)\frac{1}{2}\left[\left[E^{Hj}_t\left(\varepsilon_{H,t+1}\right)^2\right](2)-\left[E^{Hj}_t\left(\varepsilon_{F,t+1}\right)^2\right](2)\right] \\
= & \; 0
\end{aligned}
$$

Finally, using the results from the signal extraction (namely that $\left[E^{Hj}_t\left(\varepsilon_{H,t+1}\right)^3\right](3)=-\left[E^{Hj}_t\left(\varepsilon_{F,t+1}\right)^3\right](3)$ and $\left[E^{Hj}_t\left(\varepsilon_{H,t+1}\right)^2\varepsilon_{F,t+1}\right](3)=-\left[E^{Hj}_t\varepsilon_{H,t+1}\left(\varepsilon_{F,t+1}\right)^2\right](3)$), we get:

$$
\begin{aligned}
& \left[E^{Hj}_t\varepsilon^A_{t+1}\left[\varepsilon^D_{t+1}\right]^2\right](3) \\
= & \; \frac{1}{2}\left[E^{Hj}_t\left[\left[\varepsilon_{Ht+1}\right]^3+\left[\varepsilon_{Ft+1}\right]^3-\varepsilon_{Ht+1}\left[\varepsilon_{Ft+1}\right]^2-\left[\varepsilon_{Ht+1}\right]^2\varepsilon_{Ft+1}\right]\right](3) \\
= & \; 0
\end{aligned}
$$

We then get:

$$
\left[E^{Hj}_t r^A_{t+1}\left(er_{t+1}\right)^2\right](3)=\left[E_t r^A_{t+1}\right](1)\left[E_t\left(er_{t+1}\right)^2\right](2)
$$

which applies to all investors in the Home country. Following similar steps, we find that $\left[E_t^{Fj} r_{t+1}^A (er_{t+1})^2\right](3) = \left[E_t^{Hj} r_{t+1}^A (er_{t+1})^2\right](3)$.

At this stage, we can evaluate the following components of (49):

$$
\begin{aligned}
& (1-\gamma)\,\tau\left[\left[\bar{E}_t^H r_{t+1}^A\right](1) + \left[\bar{E}_t^F r_{t+1}^A\right](1)\right] \\
& -\gamma(1-\gamma)\left[\begin{array}{c} \int \frac{2z_{Hj}(0)-1}{2}\left[E_t^{Hj} r_{t+1}^A (er_{t+1})^2\right](3)\,dj \\ -\int \frac{2z_{Fj}(0)-1}{2}\left[E_t^{Fj} r_{t+1}^A (er_{t+1})^2\right](3)\,dj \end{array}\right] \\
= \; & (1-\gamma)\left[E_t r_{t+1}^A\right](1)\left[2\tau - \gamma\left[E_t (er_{t+1})^2\right](2)\, z^D(0)\right] \\
= \; & 0
\end{aligned}
$$

where we used (95)-(96) to solve for $z^D(0)$.

8.1.4 Expected $er_{t+1}(r_{t+1}^A)^2$

We next turn to $\left[\bar{E}_t^H \left(r_{t+1}^A\right)^2 er_{t+1}\right](3)$ and $\left[\bar{E}_t^F \left(r_{t+1}^A\right)^2 er_{t+1}\right](3)$. These entail cubic product, and can then be evaluated using only the linear components of (29) and (30). Notice also that we can focus on the terms that are different between the two. We again split terms between their expected and unexpected components:

$$
\begin{aligned}
& \left[E_t^{Hj}\left(r_{t+1}^A\right)^2 er_{t+1}\right](3) \\
= \; & E_t^{Hj}\left[\left[\left[E_t r_{t+1}^A\right](1) + \left[1 - r_q + r_q \alpha_{2,qA}(0)\right]\varepsilon_{t+1}^A\right]^2 er_{t+1}\right](3) \\
= \; & \left[\left[E_t r_{t+1}^A\right](1)\right]^2 E_t^{Hj}\left[er_{t+1}\right](1) \\
& + \left[1 - r_q + r_q \alpha_{2,qA}(0)\right]^2 E_t^{Hj}\left[\left[\varepsilon_{t+1}^A\right]^2 er_{t+1}\right](3) \\
& + 2\left[E_t r_{t+1}^A\right](1)\left[1 - r_q + r_q \alpha_{2,qA}(0)\right] E_t^{Hj}\left[\varepsilon_{t+1}^A er_{t+1}\right](2)
\end{aligned}
$$

As $E_t^{Hj}\left[er_{t+1}\right](1) = 0$ this becomes:

$$
\begin{aligned}
& \left[E_t^{Hj}\left(r_{t+1}^A\right)^2 er_{t+1}\right](3) \\
= \; & \left[1 - r_q + r_q \alpha_{2,qA}(0)\right]^2 E_t^{Hj}\left[\left[\varepsilon_{t+1}^A\right]^2 er_{t+1}\right](3) \\
& + 2\left[E_t r_{t+1}^A\right](1)\left[1 - r_q + r_q \alpha_{2,qA}(0)\right]\left[1 - r_q + r_q \alpha_{1,qD}(0)\right]\left[E_t^{Hj}\varepsilon_{t+1}^A \varepsilon_{t+1}^D\right](2) \\
& + 2\left[E_t r_{t+1}^A\right](1)\left[1 - r_q + r_q \alpha_{2,qA}(0)\right] r_q \alpha_{5,qD}(0)\, E_t^{Hj}\left[\varepsilon_{t+1}^A\right](1)\left[E_t^{Hj} x_{t+1}^D\right](1) \\
& - \frac{1 - r_q + r_q \alpha_{1,qD}(0)}{1 + 2\lambda^2\theta} 2\left[E_t r_{t+1}^A\right](1)\left[1 - r_q + r_q \alpha_{2,qA}(0)\right] x_t^D(1)\left[E_t^{Hj}\varepsilon_{t+1}^A\right](1)
\end{aligned}
$$

As $\left[E_t^{Hj}\varepsilon_{t+1}^A\right](1)=\left[E_t^{Hj}x_{t+1}^D\right](1)=\left[E_t^{Hj}\varepsilon_{t+1}^A\varepsilon_{t+1}^D\right](2)=0$, we get:

$$\begin{aligned}
&\left[E_t^{Hj}\left(r_{t+1}^A\right)^2 er_{t+1}\right](3)\\
=\;&\left[1-r_q+r_q\alpha_{2,qA}(0)\right]^2 E_t^{Hj}\left[\left[\varepsilon_{t+1}^A\right]^2 er_{t+1}\right](3)\\
=\;&\left[1-r_q+r_q\alpha_{2,qA}(0)\right]^2\left[1-r_q+r_q\alpha_{1,qD}(0)\right]\left[E_t^{Hj}\left[\varepsilon_{t+1}^A\right]^2\varepsilon_{t+1}^D\right](3)\\
&+\left[1-r_q+r_q\alpha_{2,qA}(0)\right]^2 r_q\alpha_{5,qD}(0)\left[E_t^{Hj}\left[\varepsilon_{t+1}^A\right]^2\right](2)\left[E_t^{Hj}x_{t+1}^D\right](1)\\
&-\frac{1-r_q+r_q\alpha_{1,qD}(0)}{1+2\lambda^2\theta}\left[1-r_q+r_q\alpha_{2,qA}(0)\right]^2 x_t^D(1)\,E_t^{Hj}\left[\left(\varepsilon_{t+1}^A\right)^2\right](2)
\end{aligned}$$

From the signal extraction results, we get:

$$\begin{aligned}
E_t^{Hj}\left[\left(\varepsilon_{t+1}^A\right)^2\right](2) &= \frac{1}{2}\sigma_a^2\\
\left[E_t^{Hj}\left[\varepsilon_{t+1}^A\right]^2\varepsilon_{t+1}^D\right](3) &= \frac{1}{4}E_t^{Hj}\left[\left(\varepsilon_{H,t+1}\right)^3-\left(\varepsilon_{F,t+1}\right)^3-\varepsilon_{H,t+1}\left(\varepsilon_{F,t+1}\right)^2+\left(\varepsilon_{H,t+1}\right)^2\left(\varepsilon_{F,t+1}\right)\right](3)\\
&= \frac{1}{2\left(1+2\lambda^2\theta\right)}\sigma_a^2 x_t^D(1)
\end{aligned}$$

Therefore we get:

$$\left[E_t^{Hj}\left(r_{t+1}^A\right)^2 er_{t+1}\right](3)=0$$

The expectation is the same for all investors in both countries.

8.1.5 Expected $er_{t+1}r_{t+1}^A$

We next turn to $\left[\bar{E}_t^H r_{t+1}^A er_{t+1}\right](3)$ and $\left[\bar{E}_t^F r_{t+1}^A er_{t+1}\right](3)$. Notice also that we can focus on the terms that are different between the two. (29) and (30) entail linear and quadratic terms, so their product entails quadratic, cubic, and higher order exponents. As we focus on the third-order terms, we limit ourselves to the quadratic and cubic terms, and write:

$$\begin{aligned}
&r_{t+1}^A er_{t+1}\\
=\;&\begin{bmatrix}(1-r_q)\left[a_{t+1}^A-\omega k_{t+1}^A\right]\\+r_q q_{t+1}^A-q_t^A\end{bmatrix}\begin{bmatrix}(1-r_q)\left[a_{t+1}^D-\omega k_{t+1}^D\right]\\+r_q q_{t+1}^D-q_t^D\end{bmatrix}\\
&+\frac{r_q(1-r_q)}{2}\begin{bmatrix}\left[q_{t+1}^A-a_{t+1}^A+\omega k_{t+1}^A\right]^2\\+\frac{1}{4}\left[q_{t+1}^D-a_{t+1}^D+\omega k_{t+1}^D\right]^2\end{bmatrix}\left[\begin{bmatrix}(1-r_q)\left[a_{t+1}^D-\omega k_{t+1}^D\right]\\+r_q q_{t+1}^D-q_t^D\end{bmatrix}\right]\\
&+r_q(1-r_q)\begin{bmatrix}(1-r_q)\left[a_{t+1}^A-\omega k_{t+1}^A\right]\\+r_q q_{t+1}^A-q_t^A\end{bmatrix}\begin{bmatrix}q_{t+1}^A-a_{t+1}^A\\+\omega k_{t+1}^A\end{bmatrix}\begin{bmatrix}q_{t+1}^D-a_{t+1}^D\\+\omega k_{t+1}^D\end{bmatrix}
\end{aligned}$$

We are only interested to the differences across agents' on their expectations of this expression. The cubic products entail only the linear part of (29) and (30), for which there is no disagreement as agents agree on the dynamics of the state variables and the first order expected innovations. We can therefore focus on:

$$
\begin{aligned}
& r_{t+1}^{A} er_{t+1} \\
= & \left[\begin{array}{c} (1-r_q)\left[a_{t+1}^{A}-\omega k_{t+1}^{A}\right] \\ +r_q q_{t+1}^{A}-q_t^{A} \end{array}\right]\left[\begin{array}{c} (1-r_q)\left[a_{t+1}^{D}-\omega k_{t+1}^{D}\right] \\ +r_q q_{t+1}^{D}-q_t^{D} \end{array}\right] \\
= & \left[\begin{array}{c} \left[(1-r_q)\left[I_2-\omega I_4\right]+r_q \alpha_{qA}\right] S_{t+1}+r_q \alpha_{5,qA} x_{t+1}^{D} \\ +r_q\left[S_{t+1}^{\prime} A_{qA} S_{t+1}+\beta_{qA} S_{t+1} x_{t+1}^{D}+\mu_{qA}\left(x_{t+1}^{D}\right)^2+\kappa_{qD}\right]-q_t^{A} \end{array}\right] \\
& \times\left[\begin{array}{c} \left[(1-r_q)\left[I_1-\omega I_3\right]+r_q \alpha_{qD}\right] S_{t+1}+r_q \alpha_{5,qD} x_{t+1}^{D} \\ +r_q\left[S_{t+1}^{\prime} A_{qD} S_{t+1}+\beta_{qD} S_{t+1} x_{t+1}^{D}+\mu_{qD}\left(x_{t+1}^{D}\right)^2+\kappa_{qD}\right]-q_t^{D} \end{array}\right]
\end{aligned}
$$

We can ignore the brackets that include cross-product, as they will only enter as cubic products of linear terms, on which there is no disagreement. Similarly there is no disagreement on the x_{t+1}^{D}. Thus we focus on:

$$
\begin{aligned}
& r_{t+1}^{A} er_{t+1} \\
= & S_{t+1}^{\prime}\left[\begin{array}{c} (1-r_q)\left[I_2-\omega I_4\right] \\ +r_q \alpha_{qA} \end{array}\right]^{\prime}\left[\begin{array}{c} (1-r_q)\left[I_1-\omega I_3\right] \\ +r_q \alpha_{qD} \end{array}\right] S_{t+1} \\
& -q_t^{D}\left[(1-r_q)\left[I_2-\omega I_4\right]+r_q \alpha_{qA}\right] S_{t+1} \\
& -q_t^{A}\left[(1-r_q)\left[I_1-\omega I_3\right]+r_q \alpha_{qD}\right] S_{t+1}+q_t^{A} q_t^{D}
\end{aligned}
$$

$q_t^A q_t^D$ are publicly known at time t and there is no disagreement on them. There is also no disagreement on the first- and second-orders of S_{t+1}, as seen from (73) and (79). Notice that there is disagreement on the second order innovations in S_{t+1}, as seen from (80), but this disagreement adds up to zero across agents of a given country.

Therefore there are no sources of disagreements between Home and Foreign aggregate third-order expectations of $r_{t+1}^{A} er_{t+1}$, hence $\left[\bar{E}_t^{H} r_{t+1}^{A} er_{t+1}\right](3)-\left[\bar{E}_t^{F} r_{t+1}^{A} er_{t+1}\right](3)=0$.

8.2 Cross-country difference

Using our results so far, (49) simplifies to:

$$\begin{aligned}
& \gamma z_t^D(1)\left[E_t\left(er_{t+1}\right)^2\right](2) \\
= \; & \left[\bar{E}_t^H er_{t+1}\right](3)-\left[\bar{E}_t^F er_{t+1}\right](3) \\
& -\gamma\left[\int \frac{2 z_{Hj}(0)-1}{2}\left[E_t^{Hj}\left(er_{t+1}\right)^2\right](3)\, dj-\int \frac{2 z_{Fj}(0)-1}{2}\left[E_t^{Fj}\left(er_{t+1}\right)^2\right](3)\, dj\right]
\end{aligned} \tag{114}$$

The first-order difference in portfolio shares reflects different expectations in third order expected excess returns in the two countries, different changes in financial frictions, as well as the expectations on time varying second moments, the latter being fully summarized by the third-order terms of the variance of excess returns.

Intuitively, the difference in portfolio shares reflect expected excess returns and frictions, scaled by the variance of the excess returns. The terms above show that movements in each of these leads to variations in the first-order difference in portfolio shares.

8.2.1 General expression for $\left[E_t^{Hj}\left(er_{t+1}\right)^2\right](3)$

The last step in the analysis is to solve for the last row in (114). We start by re-writing (30) as:

$$\begin{aligned}
er_{t+1} = \; & \left(1-r_q\right)\left[\rho a_t^D+\varepsilon_{t+1}^D\right]-\left[\omega\left(1-r_q\right)\alpha_{kD}+\alpha_{qD}\right] S_t \\
& -\left[\omega\left(1-r_q\right)\alpha_{5,kD}+\alpha_{5,qD}\right] x_t^D+r_q \alpha_{qD} S_{t+1}+r_q \alpha_{5,qD} x_{t+1}^D \\
& -\left(1-r_q\right)\left[\kappa_{qD}+\omega \kappa_{kD}\right] \\
& -S_t^{\prime}\left[\omega\left(1-r_q\right) A_{kD}+A_{qD}\right] S_t-\left[\omega\left(1-r_q\right)\beta_{kD}+\beta_{qD}\right] S_t x_t^D \\
& -\left[\omega\left(1-r_q\right)\mu_{kD}+\mu_{qD}\right]\left(x_t^D\right)^2 \\
& +S_{t+1}^{\prime}\Phi_{er1} S_{t+1}+\mu_{er1}\left(x_{t+1}^D\right)^2+\beta_{er1} S_{t+1} x_{t+1}^D
\end{aligned}$$

where Φ_{er1} is defined in (89), and:

$$\beta_{er1} = r_q \beta_{qD}+r_q\left(1-r_q\right)\left[\begin{array}{c}\alpha_{5,qA}\left[\alpha_{qD}-\left(I_1-\omega I_3\right)\right] \\ +\alpha_{5,qD}\left[\alpha_{qA}-\left(I_2-\omega I_4\right)\right]\end{array}\right] \tag{115}$$

$$\mu_{er1} = r_q \mu_{qD}+r_q\left(1-r_q\right) \alpha_{5,qA} \alpha_{5,qD} \tag{116}$$

Using (57), we write:

$$er_{t+1}=[lin\_er]_{t+1}+[quadr\_er]_{t+1}$$

where we used the fact that as we are only interested in the third order terms in $\left[E_t^{Hj}\left(er_{t+1}\right)^2\right](3)$ we limit ourselves to the linear and quadratic terms in er_{t+1}, defining:

$$\begin{aligned}
[lin_er]_{t+1} &= (1-r_q)\rho a_t^D + \{r_q\alpha_{qD}N_1 - [\omega(1-r_q)\alpha_{kD} + \alpha_{qD}]\} S_t \\
&\quad + \{r_q\alpha_{qD}N_3 - [\omega(1-r_q)\alpha_{5,kD} + \alpha_{5,qD}]\} x_t^D \\
&\quad + (1-r_q)\varepsilon_{t+1}^D + r_q\alpha_{qD}N_2\varepsilon_{t+1} + r_q\alpha_{5,qD}x_{t+1}^D
\end{aligned}$$

and:

$$\begin{aligned}
[quadr_er]_{t+1} &= r_q\alpha_{qD}\left[N_4\left(x_t^D\right)^2 + N_5 S_t x_t^D + \begin{pmatrix} 0 \\ 0 \\ S_t' N_6 S_t \\ S_t' N_7 S_t \end{pmatrix}\right] \\
&\quad + r_q\alpha_{qD}\kappa - (1-r_q)[\kappa_{qD} + \omega\kappa_{kD}] \\
&\quad - S_t'[\omega(1-r_q)A_{kD} + A_{qD}]S_t - \left[\omega(1-r_q)\beta_{kD} + \beta_{qD}\right] S_t x_t^D \\
&\quad - \left[\omega(1-r_q)\mu_{kD} + \mu_{qD}\right]\left(x_t^D\right)^2 \\
&\quad + S_{t+1}'\Phi_{er1}S_{t+1} + \mu_{er1}\left(x_{t+1}^D\right)^2 + \beta_{er1}S_{t+1}x_{t+1}^D
\end{aligned}$$

We therefore write:

$$\left[E_t^{Hj}\left(er_{t+1}\right)^2\right](3) = \left[E_t^{Hj}\left([lin\_er]_{t+1}\right)^2\right](3) + 2\left[E_t^{Hj}[lin\_er]_{t+1}[quadr\_er]_{t+1}\right](3) \tag{117}$$

8.2.2 The squared linear component

We evaluate the $\left[E_t^{Hj}\left([lin\_er]_{t+1}\right)^2\right](3)$ term in (117) as follows:

$$\begin{aligned}
&\left[E_t^{Hj}\left([lin_er]_{t+1}\right)^2\right](3) \\
=& \left[Var_t^{Hj}[lin_er]_{t+1}\right](3) + \left[\left[E_t^{Hj}\left([lin_er]_{t+1}\right)\right](3)\right]^2 \\
=& \left[Var_t^{Hj}[lin_er]_{t+1}\right](3) + 2\left[\left[E_t^{Hj}\left([lin_er]_{t+1}\right)\right](2)\right]\left[\left[E_t^{Hj}\left([lin_er]_{t+1}\right)\right](1)\right]
\end{aligned}$$

Notice that $\left[E_t^{Hj}\left([lin\_er]_{t+1}\right)\right](1)$ is simply the first order expected excess return, which is zero for all investors. The variance of terms known at time t is zero, hence we write:

$$\begin{aligned}
& \left[Var_t^{Hj}\left[lin_er\right]_{t+1}\right](3) \\
= & \left[Var_t^{Hj}\left[\left(1-r_q\right)\varepsilon_{t+1}^D + r_q\alpha_{qD}N_2\varepsilon_{t+1} + r_q\alpha_{5,qD}x_{t+1}^D\right]\right](3) \\
= & \left[Var_t^{Hj}\left[\left[1-r_q+r_q\alpha_{1,qD}\left(0\right)\right]\varepsilon_{t+1}^D + r_q\alpha_{5,qD}\left(0\right)x_{t+1}^D\right]\right](3)
\end{aligned}$$

where we used (98) and (99). As ε_{t+1}^D and x_{t+1}^D are independent, this becomes:

$$\begin{aligned}
& \left[Var_t^{Hj}\left[lin_er\right]_{t+1}\right](3) \\
= & \left[1-r_q+r_q\alpha_{1,qD}\left(0\right)\right]^2\left[Var_t^{Hj}\left[\varepsilon_{t+1}^D\right]\right](3) \\
& +\left(r_q\alpha_{5,qD}\left(0\right)\right)^2\left[Var_t^{Hj}\left[x_{t+1}^D\right]\right](3)
\end{aligned}$$

As shocks are exactly limited to second-order components, $\left[Var_t^{Hj}\left[x_{t+1}^D\right]\right](3)=0$. In the signal extraction we showed that the investors' inferences of the variances of productivity innovation do not have third-order components: $\left[Var_t^{Hj}\left[\varepsilon_{t+1}^D\right]\right](3)=0$. Therefore we get:

$$\left[E_t^{Hj}\left(\left[lin\_er\right]_{t+1}\right)^2\right](3) = \left[Var_t^{Hj}\left[lin\_er\right]_{t+1}\right](3) = 0$$

8.2.3 The linear-quadratic component

We now turn to $\left[E_t^{Hj}\left[lin\_er\right]_{t+1}\left[quadr\_er\right]_{t+1}\right](3)$ in (117). As it is a cubic products, it only involves zero-order coefficients. From the zero-order solution we write:

$$\begin{aligned}
\left[lin_er\right]_{t+1} = & \left(1-r_q\right)\rho a_t^D + \left[\begin{array}{c} r_q\left[\rho\alpha_{1,qD}\left(0\right)+\alpha_{3,qD}\left(0\right)\alpha_{1,kD}\left(0\right)\right] \\ -\left[\omega\left(1-r_q\right)\alpha_{1,kD}\left(0\right)+\alpha_{1,qD}\left(0\right)\right]\end{array}\right]a_t^D \\
& +\left[\begin{array}{c} r_q\left[\alpha_{3,qD}\left(0\right)\alpha_{3,kD}\left(0\right)\right] \\ -\left[\omega\left(1-r_q\right)\alpha_{3,kD}\left(0\right)+\alpha_{3,qD}\left(0\right)\right]\end{array}\right]k_t^D \\
& +\left\{\begin{array}{c} r_q\alpha_{3,qD}\left(0\right)\alpha_{5,kD}\left(0\right) \\ -\omega\left(1-r_q\right)\alpha_{5,kD}\left(0\right)-\alpha_{5,qD}\left(0\right)\end{array}\right\}x_t^D \\
& +\left[1-r_q+r_q\alpha_{1,qD}\left(0\right)\right]\varepsilon_{t+1}^D + r_q\alpha_{5,qD}\left(0\right)x_{t+1}^D
\end{aligned}$$

Using (70), (72) and (71) this becomes:

$$\left[lin\_er\right]_{t+1} = -\frac{1-r_q+r_q\alpha_{1,qD}\left(0\right)}{1+2\lambda^2\theta}x_t^D + \left[1-r_q+r_q\alpha_{1,qD}\left(0\right)\right]\varepsilon_{t+1}^D + r_q\alpha_{5,qD}\left(0\right)x_{t+1}^D$$

Notice that for all investors this is zero in expected value, as first-order expected excess returns are zero. We can therefore ignore the terms in $[quadr\_er]_{t+1}$ that are known at time t, as they interact with $\left[E_t^{Hj}[lin\_er]_{t+1}\right](1) = 0$ in $\left[E_t^{Hj}[lin\_er]_{t+1}[quadr\_er]_{t+1}\right](3)$. We therefore focus on:

$$[quadr\_er]_{t+1} = S'_{t+1}\Phi_{er1}S_{t+1} + \mu_{er1}\left(x_{t+1}^D\right)^2 + \beta_{er1}S_{t+1}x_{t+1}^D$$

which leads to:

$$\begin{aligned} & \left[E_t^{Hj}[lin\_er]_{t+1}[quadr\_er]_{t+1}\right](3) \\ = & -\frac{1-r_q+r_q\alpha_{1,qD}(0)}{1+\lambda^2\theta(1-\rho_\tau)}x_t^D(1)\left[E_t^{Hj}\left[S'_{t+1}\Phi_{er1}S_{t+1} + \mu_{er1}\left(x_{t+1}^D\right)^2 + \beta_{er1}S_{t+1}x_{t+1}^D\right]\right](2) \\ & +r_q\alpha_{5,qD}(0)\left[E_t^{Hj}x_{t+1}^D\left[S'_{t+1}\Phi_{er1}S_{t+1} + \mu_{er1}\left(x_{t+1}^D\right)^2 + \beta_{er1}S_{t+1}x_{t+1}^D\right]\right](3) \\ & +\left[1-r_q+r_q\alpha_{1,qD}(0)\right]\left[E_t^{Hj}\varepsilon_{t+1}^D\left[S'_{t+1}\Phi_{er1}S_{t+1} + \mu_{er1}\left(x_{t+1}^D\right)^2 + \beta_{er1}S_{t+1}x_{t+1}^D\right]\right](3) \end{aligned} \tag{118}$$

We start with the first term on the right-hand side of (118). Using (58) and the fact that S_{t+1} and x_{t+1}^D are independent, we get: $\left[E_t^{Hj}S_{t+1}x_{t+1}^D\right](2) = 0$. Therefore the term becomes:

$$-\frac{1-r_q+r_q\alpha_{1,qD}(0)}{1+2\lambda^2\theta}x_t^D(1)\left[\left[E_t^{Hj}S'_{t+1}\Phi_{er1}S_{t+1}\right](2) + \mu_{er1}2\left[1+\lambda^2\theta(1-\rho_\tau)\right]\sigma_a^2\right]$$

where $\left[E_t^{Hj}S'_{t+1}\Phi_{er1}S_{t+1}\right](2)$ can be computed following (81).

Turning to the second term on the right-hand side of (118), we use (58) to write: $\left[E_t^{Hj}x_{t+1}^D\left[S'_{t+1}\Phi_{er1}S_{t+1}\right]\right](3) = \left[E_t^{Hj}x_{t+1}^D\right](1)\left[E_t^{Hj}S'_{t+1}\Phi_{er1}S_{t+1}\right](2) = 0$. Furthermore, using (58), we get:

$$\left[E_t^{Hj}\left(x_{t+1}^D\right)^3\right] = \left[E_t^{Hj}\left(x_{t+1}^D\right)\right]^3 + 3\left[E_t^{Hj}\left(x_{t+1}^D\right)\right]Var_t^{Hj}\left(x_{t+1}^D\right) = 0$$

Therefore the term becomes:

$$\begin{aligned} & r_q\alpha_{5,qD}(0)\beta_{er1}\left[E_t^{Hj}\left(x_{t+1}^D\right)^2\right](2)\left[E_t^{Hj}S_{t+1}\right](1) \\ = & \; r_q\alpha_{5,qD}(0)\beta_{er1}2\left(1+2\lambda^2\theta\right)\sigma_a^2\begin{bmatrix} N_1(0)S_t(1) \\ +\left[N_2\frac{1}{2\left(1+2\lambda^2\theta\right)}\iota + N_3(0)\right]x_t^D(1)\end{bmatrix} \end{aligned}$$

We now turn to the last term on the right-hand side of (118). Using (58), we write: $\left[E_t^{Hj}\varepsilon_{t+1}^D\beta_{er1}S_{t+1}x_{t+1}^D\right](3) = \left[E_t^{Hj}\varepsilon_{t+1}^D\beta_{er1}S_{t+1}\right](2)\left[E_t^{Hj}x_{t+1}^D\right](1) = 0$. Furthermore:

$$\left[E_t^{Hj}\varepsilon_{t+1}^D\left(x_{t+1}^D\right)^2\right](3) = \left[E_t^{Hj}\left(x_{t+1}^D\right)^2\right](2)\left[E_t^{Hj}\varepsilon_{t+1}^D\right](1) = 2\sigma_a^2x_t^D(1)$$

Before proceeding, we write two useful expectations, which are the same for any investors in any country. First:

$$\begin{aligned}
&\left[E_t^{Hj}\varepsilon_{t+1}\varepsilon_{t+1}^{D}\right](2) \\
=&\left|\begin{array}{c}\left[E_t^{Hj}\left(\varepsilon_{H,t+1}\right)^2\right](2)-\left[E_t^{Hj}\varepsilon_{H,t+1}\varepsilon_{F,t+1}\right](2)\\ \left[E_t^{Hj}\varepsilon_{H,t+1}\varepsilon_{F,t+1}\right](2)-\left[E_t^{Hj}\left(\varepsilon_{F,t+1}\right)^2\right](2)\end{array}\right| \\
=&\ \iota\left[\frac{1}{2\left(1+2\lambda^2\theta\right)^2}\left(x_t^D\right)^2+\frac{2\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2\right]
\end{aligned}$$

where $\iota=|1,-1|'$. Second, using the definition of $\tilde{\Phi}_{er1}=N_2'\Phi_{er1}N_2$ similar to (81):

$$\begin{aligned}
&\left[E_t^{Hj}\varepsilon_{t+1}'N_2'\Phi_{er1}N_2\varepsilon_{t+1}\varepsilon_{t+1}^{D}\right](3) \\
=&\left[E_t^{Hj}\varepsilon_{t+1}'\left|\begin{array}{cc}\left[\tilde{\Phi}_{er1}\right]_{11} & \left[\tilde{\Phi}_{er1}\right]_{12}\\ \left[\tilde{\Phi}_{er1}\right]_{21} & \left[\tilde{\Phi}_{er1}\right]_{22}\end{array}\right|\varepsilon_{t+1}\varepsilon_{t+1}^{D}\right](3) \\
=&\left[\tilde{\Phi}_{er1}\right]_{11}E_t^{Hj}\left[\left(\varepsilon_{H,t+1}\right)^3\right](3)+\left(\left[\tilde{\Phi}_{er1}\right]_{12}+\left[\tilde{\Phi}_{er1}\right]_{21}-\left[\tilde{\Phi}_{er1}\right]_{11}\right)E_t^{Hj}\left[\left(\varepsilon_{H,t+1}\right)^2\varepsilon_{F,t+1}\right](3) \\
&+\left(\left[\tilde{\Phi}_{er1}\right]_{22}-\left[\tilde{\Phi}_{er1}\right]_{12}-\left[\tilde{\Phi}_{er1}\right]_{21}\right)E_t^{Hj}\left[\varepsilon_{H,t+1}\left(\varepsilon_{F,t+1}\right)^2\right](3)-\left[\tilde{\Phi}_{er1}\right]_{22}E_t^{Hj}\left[\left(\varepsilon_{F,t+1}\right)^3\right](3)
\end{aligned}$$

Using the results from the signal extraction, this becomes:

$$\begin{aligned}
&\left[E_t^{Hj}\varepsilon_{t+1}'N_2'\Phi_{er1}N_2\varepsilon_{t+1}\varepsilon_{t+1}^{D}\right](3) \\
=&\left(\left[\tilde{\Phi}_{er1}\right]_{11}+\left[\tilde{\Phi}_{er1}\right]_{22}\right)E_t^{Hj}\left[\left(\varepsilon_{H,t+1}\right)^3\right](3) \\
&+\left(2\left(\left[\tilde{\Phi}_{er1}\right]_{12}+\left[\tilde{\Phi}_{er1}\right]_{21}\right)-\left(\left[\tilde{\Phi}_{er1}\right]_{11}+\left[\tilde{\Phi}_{er1}\right]_{22}\right)\right)E_t^{Hj}\left[\left(\varepsilon_{H,t+1}\right)^2\varepsilon_{F,t+1}\right](3) \\
=&\ 2\left(\begin{array}{c}\left[\tilde{\Phi}_{er1}\right]_{11}+\left[\tilde{\Phi}_{er1}\right]_{22}\\ -\left[\tilde{\Phi}_{er1}\right]_{12}-\left[\tilde{\Phi}_{er1}\right]_{21}\end{array}\right)\left(\frac{1}{2\left(1+2\lambda^2\theta\right)}x_t^D\right)^3 \\
&+\left[\begin{array}{c}\left(\left[\tilde{\Phi}_{er1}\right]_{11}+\left[\tilde{\Phi}_{er1}\right]_{22}\right)\frac{1+4\lambda^2\theta}{1+2\lambda^2\theta}\\ +\left(\left[\tilde{\Phi}_{er1}\right]_{12}+\left[\tilde{\Phi}_{er1}\right]_{21}\right)\frac{1-4\lambda^2\theta}{1+2\lambda^2\theta}\end{array}\right]\frac{1}{2\left(1+2\lambda^2\theta\right)}x_t^D\sigma_a^2
\end{aligned}$$

Using these results and (57), we write:

$$
\begin{aligned}
& \left[E_t^{Hj}\varepsilon_{t+1}^D S_{t+1}' \Phi_{er1} S_{t+1}\right](3) \\
= & \left[N_1 S_t + N_3 x_t^D\right]' \Phi_{er1}\left[N_1 S_t + N_3 x_t^D\right]\left[E_t^{Hj}\varepsilon_{t+1}^D\right](1) \\
& + \left[N_1 S_t + N_3 x_t^D\right]' \left(\Phi_{er1} + \Phi_{er1}'\right) N_2 \left[E_t^{Hj}\varepsilon_{t+1}\varepsilon_{t+1}^D\right](2) \\
& + \left[E_t^{Hj}\varepsilon_{t+1}' N_2' \Phi_{er1} N_2 \varepsilon_{t+1}\varepsilon_{t+1}^D\right](3) \\
= & \left[N_1 S_t + N_3 x_t^D\right]' \Phi_{er1}\left[N_1 S_t + N_3 x_t^D\right] \frac{1}{1+2\lambda^2\theta} x_t^D \\
& + \left[N_1 S_t + N_3 x_t^D\right]' \left(\Phi_{er1} + \Phi_{er1}'\right) N_2 \iota \left[\frac{1}{2\left(1+2\lambda^2\theta\right)^2}\left(x_t^D\right)^2 + \frac{2\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2\right] \\
& +2 \begin{pmatrix} \left[\tilde{\Phi}_{er1}\right]_{11} + \left[\tilde{\Phi}_{er1}\right]_{22} \\ -\left[\tilde{\Phi}_{er1}\right]_{12} - \left[\tilde{\Phi}_{er1}\right]_{21} \end{pmatrix} \left(\frac{1}{2\left(1+2\lambda^2\theta\right)} x_t^D\right)^3 \\
& + \begin{bmatrix} \left(\left[\tilde{\Phi}_{er1}\right]_{11} + \left[\tilde{\Phi}_{er1}\right]_{22}\right)\frac{1+4\lambda^2\theta}{1+2\lambda^2\theta} \\ +\left(\left[\tilde{\Phi}_{er1}\right]_{12} + \left[\tilde{\Phi}_{er1}\right]_{21}\right)\frac{1-4\lambda^2\theta}{1+2\lambda^2\theta} \end{bmatrix} \frac{1}{2\left(1+2\lambda^2\theta\right)} x_t^D \sigma_a^2
\end{aligned}
$$

Putting our results together, (118) becomes

$$
\begin{aligned}
& \left[E_t^{Hj}\left[lin\_er\right]_{t+1}\left[quadr\_er\right]_{t+1}\right](3) \\
= & \Phi_{LQX} x_t^D (1)\sigma_a^2 + \Phi_{LQS} S_t (1)\sigma_a^2
\end{aligned}
\tag{119}
$$

where we used $N_2\iota = 2\left(I_1\right)'$, the solution for $\alpha_{5,qD}(0)$ in (71), (81), and:

$$
\begin{aligned}
\Phi_{LQX} &= \frac{1-r_q+r_q\alpha_{1,qD}(0)}{1+2\lambda^2\theta}\begin{bmatrix} \begin{bmatrix} \left[\tilde{\Phi}_{er1}\right]_{11} + \left[\tilde{\Phi}_{er1}\right]_{22} \\ -\left(\left[\tilde{\Phi}_{er1}\right]_{12} + \left[\tilde{\Phi}_{er1}\right]_{21}\right) \end{bmatrix}\frac{2\lambda^2\theta}{1+2\lambda^2\theta} \\ +4\lambda^2\theta I_1\left(\Phi_{er1}+\Phi_{er1}'\right) N_3(0) \\ +\frac{r_q 2\left(1+2\lambda^2\theta\right)}{1+\left[(1-r_q)\omega - r_q\alpha_{3,qD}(0)\right]\frac{1}{\xi}}\beta_{er1}\left[\frac{1}{1+2\lambda^2\theta}\left(I_1\right)' + N_3(0)\right] \end{bmatrix} \\
\Phi_{LQS} &= 2\frac{1-r_q+r_q\alpha_{1,qD}(0)}{1+2\lambda^2\theta}\begin{bmatrix} I_1\left(\Phi_{er1}+\Phi_{er1}'\right)2\lambda^2\theta \\ +\frac{r_q\left(1+2\lambda^2\theta\right)}{1+\left[(1-r_q)\omega - r_q\alpha_{3,qD}(0)\right]\frac{1}{\xi}}\beta_{er1} \end{bmatrix} N_1(0)
\end{aligned}
$$

8.2.4 Further simplifications

The coefficients in (119) can be substantially simplified. We start with the $A_{qD}(0)$ matrix which solves (103). $A_{qD}(other)$ consists of $A_{er}(other)$ and $A_{kD}(other)$.

$A_{kD}\,(other)$ is given by (78):

$$\begin{aligned} A_{kD}\,(other) &= \frac{\xi-1}{\xi}\begin{bmatrix} \alpha_{1,qD}\,(0)\,I_1 \\ +\alpha_{3,qD}\,(0)\,I_3 \end{bmatrix}'\begin{bmatrix} \alpha_{2,kA}\,(0)\,I_2 \\ +\left(\alpha_{4,kA}\,(0)-1\right)I_4 \end{bmatrix} \\ &= \frac{\xi-1}{\xi}\begin{vmatrix} \alpha_{1,qD}\,(0) \\ 0 \\ \alpha_{3,qD}\,(0) \\ 0 \end{vmatrix}\begin{vmatrix} 0 & \alpha_{2,kA}\,(0) & 0 & \alpha_{4,kA}\,(0)-1 \end{vmatrix} \\ &= \begin{vmatrix} 0 & x & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & x & 0 & x \\ 0 & 0 & 0 & 0 \end{vmatrix} \end{aligned}$$

where the x's denote non-zero elements (which need not be equal to each other). $A_{er}\,(other)$ is given by (100). Using the solution for $N_1\,(0)$, we can show that:

$$A_{er}\,(other) = \begin{vmatrix} 0 & 0 & 0 & 0 \\ x & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ x & 0 & x & 0 \end{vmatrix}$$

It follows that the form of $A_{qD}(other)$ is:

$$A_{qD}(other) = \begin{vmatrix} 0 & x & 0 & x \\ x & 0 & x & 0 \\ 0 & x & 0 & x \\ x & 0 & x & 0 \end{vmatrix} \tag{120}$$

Notice that if a matrix A is of the form (120), its transposed is also of that form. In addition $AN_1\,(0)$ is also of the form (120), as is $N_1\,(0)'\,AN_1\,(0)$. As $A_{qD}(other)$ is of the form (120), (103) then implies that $A_{qD}\,(0)$ is also of the form (120).

We next turn to Φ_{er1}. Using (89) we write:

$$\begin{aligned} \Phi_{er1} &= r_q A_{qD}\,(0) + r_q\,(1-r_q)\left[\alpha_{qA}\,(0) - (I_2-\omega I_4)\right]'\left[\alpha_{qD}\,(0) - (I_1-\omega I_3)\right] \\ &= r_q A_{qD}\,(0) + \begin{vmatrix} 0 & 0 & 0 & 0 \\ x & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ x & 0 & x & 0 \end{vmatrix} \end{aligned}$$

As $A_{qD}(0)$ is of the form (120), Φ_{er1} is also of that form. Now recall that $\tilde{\Phi}_{er1} = N_2' \Phi_{er1} N_2$, which implies:

$$\begin{aligned}
\left[\tilde{\Phi}_{er1}\right]_{11} &= [\Phi_{er1}]_{11} + \frac{[\Phi_{er1}]_{12}}{2} + \frac{1}{2}\left[[\Phi_{er1}]_{21} + \frac{[\Phi_{er1}]_{22}}{2}\right] \\
\left[\tilde{\Phi}_{er1}\right]_{12} &= -[\Phi_{er1}]_{11} + \frac{[\Phi_{er1}]_{12}}{2} + \frac{1}{2}\left[-[\Phi_{er1}]_{21} + \frac{[\Phi_{er1}]_{22}}{2}\right] \\
\left[\tilde{\Phi}_{er1}\right]_{21} &= -[\Phi_{er1}]_{11} - \frac{[\Phi_{er1}]_{12}}{2} + \frac{1}{2}\left[[\Phi_{er1}]_{21} + \frac{[\Phi_{er1}]_{22}}{2}\right] \\
\left[\tilde{\Phi}_{er1}\right]_{22} &= [\Phi_{er1}]_{11} - \frac{[\Phi_{er1}]_{12}}{2} + \frac{1}{2}\left[-[\Phi_{er1}]_{21} + \frac{[\Phi_{er1}]_{22}}{2}\right]
\end{aligned}$$

We can then write:

$$\begin{aligned}
& \left(\left[\tilde{\Phi}_{er1}\right]_{11} + \left[\tilde{\Phi}_{er1}\right]_{22}\right) - \left(\left[\tilde{\Phi}_{er1}\right]_{12} + \left[\tilde{\Phi}_{er1}\right]_{21}\right) \\
&= 4\,[\Phi_{er1}]_{11} \\
&= 0
\end{aligned}$$

where $[\Phi_{er1}]_{11} = 0$ because Φ_{er1} is of the form (120).

As Φ_{er1} is of the form (120), $\Phi_{er1} + \Phi_{er1}'$ is also of that form. Using our solution for $N_3(0)$, this implies that $(\Phi_{er1} + \Phi_{er1}')\, N_3(0)$ is a 4x1 vector with a non-zero element only in the third row, hence

$$I_1 \left(\Phi_{er1} + \Phi_{er1}'\right) N_3(0) = 0$$

Also, $(\Phi_{er1} + \Phi_{er1}')\, N_1(0)$ is of the form (120), hence $I_1 (\Phi_{er1} + \Phi_{er1}')\, N_1(0)$ is a 1x4 vector of with non-zero elements in the second and fourth columns.

We now turn to the β_{er1} vector. As $\alpha_{5,qA}(0) = 0$, (115) implies that β_{er1} consists of $r_q \beta_{qD}(0)$ and a 1x4 vector of with non-zero elements in the second and fourth columns. (106) shows that $\beta_{qD}(0)$ is a combination of $\beta_{kD}(other)$ and $\beta_{er}(other)$, where:

$$\begin{aligned}
\beta_{kD}(other) &= \frac{\xi - 1}{\xi} \alpha_{5,qD}(0) \left[\alpha_{2,kA}(0)\, I_2 + \left(\alpha_{4,kA}(0) - 1\right) I_4\right] \\
\beta_{er}(other) &= \left[\frac{1}{1 + 2\lambda^2\theta} (I_1)' + N_3(0)\right]' \left(\Phi_{er1} + \Phi_{er1}'\right) N_1(0)
\end{aligned}$$

$\beta_{kD}(other)$ is clearly a 1x4 vector of with non-zero elements in the second and fourth columns. As $(\Phi_{er1} + \Phi_{er1}')\, N_1(0)$ is of the form (120) and $\left[\frac{1}{1+2\lambda^2\theta} I_1 + N_3(0)\right]'$

is a 1x4 vector of with non-zero elements in the first and third columns, we get that $\beta_{er}(other)$ is a 1x4 vector of with non-zero elements in the second and fourth columns.

It then follows that β_{er1} is a 1x4 vector of with non-zero elements in the second and fourth columns, and so is $\beta_{er1}N_1(0)$. This implies:

$$\beta_{er1}\left[\frac{1}{1+2\lambda^2\theta}(I_1)' + N_3(0)\right] = 0$$

Using these simplifications, (119) becomes:

$$\left[E_t^{Hj}[lin\_er]_{t+1}[quadr\_er]_{t+1}\right](3) = \Phi_{LQS}S_t(1)\sigma_a^2 \tag{121}$$

where:

$$\Phi_{LQS} = 2\frac{1-r_q+r_q\alpha_{1,qD}(0)}{1+2\lambda^2\theta}\left[\begin{array}{c} I_1\left(\Phi_{er1}+\Phi'_{er1}\right)2\lambda^2\theta \\ +\frac{r_q\left(1+2\lambda^2\theta\right)}{1+\left[(1-r_q)\omega-r_q\alpha_{3,qD}(0)\right]\frac{1}{\xi}}\beta_{er1}\end{array}\right]N_1(0)$$

As Φ_{LQS} is a 1x4 vector of with non-zero elements in the first and third columns, only the worldwide averages of the state variables ($a_t^A(1)$ and $k_t^A(1)$) enter (121), with no role for the cross-country differences of the state variables ($a_t^D(1)$ and $k_t^D(1)$).

8.2.5 Solution for the cross-country difference

Putting all our steps together, (114) becomes:

$$z_t^D(1) = \frac{\left[\bar{E}_t^H er_{t+1}\right](3) - \left[\bar{E}_t^F er_{t+1}\right](3)}{\gamma\left[E_t(er_{t+1})^2\right](2)} - z^D(0)\frac{\left[E_t(er_{t+1})^2\right](3)}{\left[E_t(er_{t+1})^2\right](2)} \tag{122}$$

where:

$$\begin{aligned}&\left[E_t(er_{t+1})^2\right](2)\\ =\;& 2\frac{\left(1-r_q+r_q\alpha_{1,qD}(0)\right)^2}{1+2\lambda^2\theta}\left[2\lambda^2\theta+\left(\frac{r_q}{1+\left[(1-r_q)\omega-r_q\alpha_{3,qD}(0)\right]\frac{1}{\xi}}\right)^2\right]\sigma_a^2\end{aligned}$$

and:

$$\left[\bar{E}_t^H er_{t+1}\right](3) - \left[\bar{E}_t^F er_{t+1}\right](3) = 2\left[1-r_q+r_q\alpha_{1,qD}(0)\right]\frac{2\lambda^2\theta\sigma_a^2}{1+2\lambda^2\theta}\frac{\sigma_{H,F}^2-\sigma_{H,H}^2}{\sigma_{H,H}^2\sigma_{H,F}^2}\varepsilon_{t+1}^A$$

8.3 Worldwide average

The worldwide average of the portfolio shares follows from the first-order component of (36):

$$4z_t^A(1) = q_t^D(1) + k_{t+1}^D(1) - z^D(0)\left[a_t^D(1) + (1-\omega)k_t^D(1)\right]$$

Using (71) and (72) we get:

$$\begin{aligned} z_t^A(1) &= \frac{1}{4}\left[\frac{1+\xi}{\xi}\alpha_{1,qD}(0) - z^D(0)\right]a_t^D(1) + \frac{1+\xi}{4\xi}\alpha_{5,qD}(0)x_t^D(1) \\ &+\frac{1}{4}\left[1+\frac{1+\xi}{\xi}\alpha_{3,qD}(0) - z^D(0)(1-\omega)\right]k_t^D(1) \end{aligned} \quad (123)$$

(123) gives the average first-order portfolio share from an asset supply perspective.

To obtain the share from an asset demand perspective, we use (48). From our results above, we know that $\left[\bar{E}_t^H r_{t+1}^A\right](1) = \left[\bar{E}_t^F r_{t+1}^A\right](1)$ and $\left[E_t^{Hj}\left(r_{t+1}^A\right)^2 er_{t+1}\right](3) = 0$ for all Home and Foreign investors. In addition, $\left[E_t^{Hj}\left(er_{t+1}\right)^2\right](3)$, $\left[E_t^{Hj}r_{t+1}^A\left(er_{t+1}\right)^2\right](3)$ and $\left[E_t^{Hj}r_{t+1}^A er_{t+1}\right](3)$ are the same for all agents. As $z^A(0) = 0.5$, (48) becomes:

$$\begin{aligned} z_t^A(1) &= \frac{\left[\bar{E}_t^H er_{t+1}\right](3) + \left[\bar{E}_t^F er_{t+1}\right](3)}{2\gamma\left[E_t\left(er_{t+1}\right)^2\right](2)} + \frac{\tau_t^D(3)}{2\gamma\left[E_t\left(er_{t+1}\right)^2\right](2)} \\ &+\frac{1-\gamma}{\gamma}\frac{\left[E_t r_{t+1}^A er_{t+1}\right](3)}{\left[E_t\left(er_{t+1}\right)^2\right](2)} \end{aligned}$$

As $r_{Ht+1} = r_{t+1}^A + 0.5er_{t+1}$ and $r_{Ft+1} = r_{t+1}^A - 0.5er_{t+1}$, we get $\left[var_t\left(r_{Ht+1}\right)\right](3) - \left[var_t\left(r_{Ft+1}\right)\right](3) = 2\left[E_t r_{t+1}^A er_{t+1}\right](3)$, which implies:

$$\begin{aligned} z_t^A(1) &= \frac{\left[\bar{E}_t^A er_{t+1}\right](3)}{\gamma\left[E_t\left(er_{t+1}\right)^2\right](2)} + \frac{\tau_t^D(3)}{2\gamma\left[E_t\left(er_{t+1}\right)^2\right](2)} \\ &+\frac{1-\gamma}{\gamma}\frac{\left[var_t\left(r_{Ht+1}\right)\right](3) - \left[var_t\left(r_{Ft+1}\right)\right](3)}{2\left[E_t\left(er_{t+1}\right)^2\right](2)} \end{aligned} \quad (124)$$

9 Algorithm for numerical solution

The numerical solution for the model can be computed following the following steps.

1. Pick values for ρ, ω, δ, ε, σ_a^2, σ_{HH}^2, σ_{HF}^2, θ, $\tau(2)$, ρ_τ.

2. Get the zero order solution from (25), which gives $\bar{c}$ and r_q.

3. Solve for $\alpha_{1-5,qD}(0)$, $\alpha_{1-5,kD}(0)$, $\alpha_{1-5,cD}(0)$ from (68), (71), (72).

4. Solve λ from (112).

5. Solve for $\alpha_{1-5,qA}(0)$, $\alpha_{1-5,kA}(0)$, $\alpha_{1-5,cA}(0)$ from (67), (66), (65).

6. Solve $\left[E_t\left(er_{t+1}\right)^2\right](2)$ from (60), and $\left[\bar{E}_t^H er_{t+1}\right](3) - \left[\bar{E}_t^F er_{t+1}\right](3)$ from (113).

7. Solve for zero-order portfolios from (95) and (96), along with the cross-sectional dispersion of portfolios $\int\left(z_{Hj}(0) - z_H(0)\right)^2 dj$, a useful measure of the consequence of info dispersion.

8. Solve for A_{cA} from (102), β_{cA} from (105), μ_{cA} from (107).

9. Solve for A_{qD} from (104), β_{qD} from (106), μ_{qD} from (108).

10. Solve for N from (74).

11. Solve for the A's, β's and μ's for q^A (from (76)) k^A (from (77)), k^D (from (78)).

12. Solve for Φ_{ra1} and Φ_{ra2} from (82) and (88).

13. Solve for Φ_{er1} from (89) and β_{er1} from (115).

14. Solve the Φ_{LQ}'s from (119).

15. Solve for $z_t^D(1)$ from (122), and $z_t^A(1)$ from (123).

10 Balance of Payments Accounting

10.1 National savings and investment

In period t the old Home agents enter the period with the following quantities of equities (we abstract from j indexes as we focus on first-order aggregates):

$$G_{H,t-1}^H = \frac{z_{H,t-1}\left(W_{H,t-1} - C_{y,t-1}^H\right)}{Q_{H,t-1}} \quad ; \quad G_{F,t-1}^H = \frac{\left(1 - z_{H,t-1}\right)\left(W_{H,t-1} - C_{y,t-1}^H\right)}{Q_{F,t-1}}$$

The consumption of old agents is the total return on their portfolio. The income of old agents is the dividend stream they receive, while capital gains and losses are not counted as income streams in national accounts. We also consider that income is net of depreciation. The savings of the old Home agents are then (we ignore the iceberg cost on holdings of Foreign equity as it represents a source of income for intermediaries that is fully consumed, and thus does not affect savings):

$$\begin{aligned} S_{o,t}^{H} &= \left[(1-\omega) A_{Ht} (K_{H,t})^{-\omega} - \delta Q_{H,t}\right] G_{H,t-1}^{H} \\ &\quad + \left[(1-\omega) A_{Ft} (K_{F,t})^{-\omega} - \delta Q_{F,t}\right] G_{F,t-1}^{H} \\ &\quad - \left(R_{H,t} Q_{H,t-1} G_{H,t-1}^{H} + R_{F,t} Q_{F,t-1} G_{F,t-1}^{H}\right) \\ &= -Q_{H,t} G_{H,t-1}^{H} - Q_{F,t} G_{F,t-1}^{H} \\ &= -\left[z_{Hj,t-1} \frac{Q_{H,t}}{Q_{H,t-1}} + (1 - z_{Hj,t-1}) \frac{Q_{F,t}}{Q_{F,t-1}}\right] \left(W_{H,t-1} - C_{y,t-1}^{H}\right) \end{aligned}$$

The dissavings by old agents reflects the liquidation value of their portfolio. National savings in the Home country are:

$$\begin{aligned} S_{t}^{H} &= S_{y,t}^{H} + S_{o,t}^{H} \\ &= W_{H,t} - C_{y,t}^{H} - \left[z_{H,t-1} \frac{Q_{H,t}}{Q_{H,t-1}} + (1 - z_{H,t-1}) \frac{Q_{F,t}}{Q_{F,t-1}}\right] \left(W_{H,t-1} - C_{y,t-1}^{H}\right) \end{aligned} \tag{125}$$

Investment in the national accounts is also defined as net of depreciation:

$$I_{H,t}^{net} = I_{H,t} - \delta K_{H,t} = K_{H,t+1} - K_{H,t} \tag{126}$$

The corresponding relation for the Foreign country are:

$$S_{t}^{F} = W_{F,t} - C_{y,t}^{F} - \left[z_{F,t-1} \frac{Q_{H,t}}{Q_{H,t-1}} + (1 - z_{F,t-1}) \frac{Q_{F,t}}{Q_{F,t-1}}\right] \left(W_{F,t-1} - C_{y,t-1}^{F}\right) \tag{127}$$

$$I_{F,t}^{net} = I_{F,t} - \delta K_{F,t-1} \tag{128}$$

Using (21) and (22), the values of world savings and investment are equal: $S_{t}^{H} + S_{t}^{F} = Q_{H,t} I_{H,t}^{net} + Q_{F,t} I_{F,t}^{net}$.

The first-order component of (125) is written as:

$$\begin{aligned} s_{t}^{H}(1) &= \frac{1}{1-\bar{c}} \Delta w_{H,t}(1) - \frac{\bar{c}}{1-\bar{c}} \Delta c_{y,t}^{H}(1) - z_{H}(0) \Delta q_{H,t}(1) - (1 - z_{H}(0)) \Delta q_{F,t}(1) \\ &= \frac{1}{1-\bar{c}} \left[\Delta a_{H,t}(1) + (1-\omega) \Delta k_{H,t}(1)\right] - \frac{\bar{c}}{1-\bar{c}} \Delta c_{y,t}^{H}(1) \\ &\quad - z_{H}(0) \Delta q_{H,t}(1) - (1 - z_{H}(0)) \Delta q_{F,t}(1) \end{aligned}$$

where $s_t^H(1) = S_t^H(1)/[(1-\bar{c})\exp[w(0)]]$, and $\Delta g_t(1) = g_t(1) - g_{t-1}(1)$. Using (68) this becomes:

$$\begin{aligned}
s_t^H(1) &= \frac{1}{1-\bar{c}}\left[\Delta a_t^A(1) + (1-\omega)\Delta k_t^A(1)\right] - \frac{\bar{c}}{1-\bar{c}}\Delta c_{y,t}^A(1) \\
&\quad + \frac{1}{2}\left[\Delta a_t^D(1) + (1-\omega)\Delta k_t^D(1)\right] - \Delta q_t^A(1) - \frac{z^D(0)}{2}\Delta q_t^D(1) \\
&= \alpha_{sH}(0)\Delta S_t(1) - \frac{z^D(0)}{2}\Delta q_t^D(1) \qquad (129)
\end{aligned}$$

where:

$$\begin{aligned}
\alpha_{sH}(0) &= \left[\frac{1}{1-\bar{c}} - \frac{\bar{c}}{1-\bar{c}}\alpha_{2,cA}(0) - \alpha_{2,qA}(0)\right] I_2 + \frac{1}{2}\left[I_1 + (1-\omega) I_3\right] \\
&\quad + \left[\frac{1}{1-\bar{c}}(1-\omega) - \frac{\bar{c}}{1-\bar{c}}\alpha_{4,cA}(0) - \alpha_{4,qA}(0)\right] I_4
\end{aligned}$$

Similarly, the first-order component of (127) is:

$$s_t^F(1) = \alpha_{sF}(0)\Delta S_t(1) + \frac{z^D(0)}{2}\Delta q_t^D(1) \qquad (130)$$

where:

$$\begin{aligned}
\alpha_{sF}(0) &= \left[\frac{1}{1-\bar{c}} - \frac{\bar{c}}{1-\bar{c}}\alpha_{2,cA}(0) - \alpha_{2,qA}(0)\right] I_2 - \frac{1}{2}\left[I_1 + (1-\omega) I_3\right] \\
&\quad + \left[\frac{1}{1-\bar{c}}(1-\omega) - \frac{\bar{c}}{1-\bar{c}}\alpha_{4,cA}(0) - \alpha_{4,qA}(0)\right] I_4
\end{aligned}$$

Taking the difference between (129) and (130) we get:

$$s_t^D(1) = \Delta a_t^D(1) + (1-\omega)\Delta k_t^D(1) - z^D(0)\Delta q_t^D(1) \qquad (131)$$

Using (8), the first-order component of (126) is:

$$i_{H,t}^{net}(1) = \Delta k_{H,t+1}(1) = \frac{1}{\xi} q_{H,t}(1) \qquad (132)$$

where $i_{H,t}^{net}(1) = I_{H,t}^{net}(1)/\exp[k(0)] = I_{H,t}^{net}(1)/[(1-\bar{c})\exp[w(0)]]$. Similarly, the first-order component of (128) is:

$$i_{F,t}^{net}(1) = \frac{1}{\xi} q_{F,t}(1) \qquad (133)$$

10.2 Capital flows

A useful measure is the passive portfolio share that combines the steady-state holdings of quantities of assets with the actual asset prices. For Home investors, we write:

$$z^p_{H,t} = \frac{z_H(0)\exp[q_{H,t}]}{z_H(0)\exp[q_{H,t}] + (1 - z_H(0))\exp[q_{F,t}]}$$

The first-order passive portfolio share is the same for all investors:

$$z^p_t(1) = \frac{1 - \left(z^D(0)\right)^2}{4} q^D_t(1) \tag{134}$$

Taking the difference between the first-order components of (21) and (22), and using (68), we write:

$$q^D_t(1) + k^D_{t+1}(1) = \left[a^D_t(1) + (1-\omega)k^D_t(1)\right] z^D(0) + 4z^A_t(1)$$

Next, we take the difference between this relation and its lagged value and use (131) and (134) to obtain:

$$i^{D,net}_t(1) - z^D(0)s^D_t(1) = 4\left[\Delta z^A_t(1) - \Delta z^p_t(1)\right]$$

Gross outflows and inflows are the change in the value of cross-border asset holdings, evaluated at current asset prices:

$$\begin{aligned} OUTFLOWS_t &= Q_{F,t}\left(G^H_{F,t} - G^H_{F,t-1}\right) \\ &= (1 - z_{H,t})\left(W_{H,t} - C^H_{y,t}\right) - \frac{Q_{F,t}}{Q_{F,t-1}}(1 - z_{H,t-1})\left(W_{H,t-1} - C^H_{y,t-1}\right) \\ INFLOWS_t &= z_{F,t}\left(W_{F,t} - C^F_{y,t}\right) - \frac{Q_{H,t}}{Q_{H,t-1}} z_{F,t-1}\left(W_{F,t-1} - C^F_{y,t-1}\right) \end{aligned}$$

Using (129) and (134), the first-order components of gross outflows is:

$$\begin{aligned} & outflows_t(1) \\ = \; & -\Delta z_{H,t}(1) + (1 - z_H(0))\frac{1}{1-\bar{c}}\left[\Delta a_{H,t}(1) + (1-\omega)\Delta k_{H,t}(1)\right] \\ & - (1 - z_H(0))\frac{\bar{c}}{1-\bar{c}}\Delta c^H_{y,t}(1) - (1 - z_H(0))\Delta q_{F,t}(1) \\ = \; & (1 - z_H(0))s^H_t(1) - \left[\Delta z^A_t(1) - \Delta z^p_t(1)\right] - \frac{1}{2}\Delta z^D_t(1) \\ = \; & (1 - z_H(0))s^H_t(1) + \frac{z^D(0)}{2}\frac{\Delta\left[var_t(er_{t+1})\right](3)}{\left[E_t(er_{t+1})^2\right](2)} \\ & - \frac{\Delta \bar{E}_t er_{t+1}(3)^{IS}}{\gamma\left[E_t(er_{t+1})^2\right](2)} - \frac{1}{2}\frac{\Delta\left[\bar{E}^H_t er_{t+1}\right](3) - \Delta\left[\bar{E}^F_t er_{t+1}\right](3)}{\gamma\left[E_t(er_{t+1})^2\right](2)} \end{aligned} \tag{135}$$

where we used (122) and we defined:

$$\Delta\bar{E}_t er_{t+1}(3)^{IS} = \frac{\gamma\left[E_t\left(er_{t+1}\right)^2\right](2)}{4}\left[i_t^D(1) - z^D(0)s_t^D(1)\right]$$

Similarly, the first-order component of gross inflows is:

$$\begin{aligned} & inflows_t\left(1\right) \\ = \;& z_F\left(0\right)s_t^F\left(1\right) + \left[\Delta z_t^A\left(1\right) - \Delta z_t^p\left(1\right)\right] - \frac{1}{2}\Delta z_t^D\left(1\right) \\ = \;& \left(1 - z_H\left(0\right)\right)s_t^F\left(1\right) + \frac{z^D\left(0\right)}{2}\frac{\Delta\left[var_t\left(er_{t+1}\right)\right](3)}{\left[E_t\left(er_{t+1}\right)^2\right](2)} \\ & + \frac{\Delta\bar{E}_t er_{t+1}(3)^{IS}}{\gamma\left[E_t\left(er_{t+1}\right)^2\right](2)} - \frac{1}{2}\frac{\Delta\left[\bar{E}_t^H er_{t+1}\right](3) - \Delta\left[\bar{E}_t^F er_{t+1}\right](3)}{\gamma\left[E_t\left(er_{t+1}\right)^2\right](2)} \end{aligned} \tag{136}$$

Combining (135) and (136) we write:

$$\begin{aligned} outflows_t\left(1\right) - inflows_t\left(1\right) &= \frac{1}{2}\left[s_t^D\left(1\right) - i_t^{D,net}\left(1\right)\right] \\ outflows_t\left(1\right) + inflows_t\left(1\right) &= \left(1 - z^D\left(0\right)\right)s_t^A\left(1\right) - \Delta z_t^D\left(1\right) \end{aligned}$$