Technical appendix for (not for publication):
The Pruned State-Space System for Non-Linear DSGE Models: Theory and Empirical Applications

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This technical appendix explans in great detail the derivations carried out in relation to our paper. In addition to the material reported in the paper, this technical appendix also provides some additional results - for instance alternative ways of computing second moments (at second and third order) and how to directly implement pruning based on the Dynare notation.

1 The class of DSGE model

We consider the class of DSGE models where the set of equilibrium conditions can be written as

$$E_t \left[\mathbf{f}\left(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{x}_t \right) \right] = \mathbf{0}. \tag{1}$$

Here, E_t is the conditional expectation given information available at time t. The vector $\mathbf{x}_t$ is the set of state variables (pre-determined variables) and has dimension $n_x \times 1$. The vector $\mathbf{y}_t$ contains the set of control variables (non pre-determined variables) and has dimension $n_y \times 1$. We also let $n \equiv n_x + n_y$.

The state vector is partitioned as $\mathbf{x}_t \equiv \begin{bmatrix} \mathbf{x}_{1,t} \\ \mathbf{x}_{2,t} \end{bmatrix}$, where $\mathbf{x}_{1,t}$ with dimension $n_{x_1} \times 1$ contains the set of endogenous state variables and $\mathbf{x}_{2,t}$ with dimension $n_{x_2} \times 1$ contains the set of exogenous state variables. Note also that $n_{x_1} + n_{x_2} = n_x$.

For the exogenous state variables we assume that

$$\mathbf{x}_{2,t+1} = \mathbf{h}\left(\mathbf{x}_{2,t}, \sigma \right) + \sigma \tilde{\boldsymbol{\eta}} \boldsymbol{\epsilon}_{t+1}, \tag{2}$$

where $\boldsymbol{\epsilon}_{t+1}$ has dimension $n_e \times 1$, and thus, $\tilde{\boldsymbol{\eta}}$ has dimension $n_{x_2} \times n_e$. We assume throughout that $\boldsymbol{\epsilon}_{t+1} \sim \mathcal{IID}\left(\mathbf{0}, \mathbf{I} \right)$, that is the innovations are identical and independent distributed with mean zero and covariance matrix $\mathbf{I}$. Further moment requirements on $\boldsymbol{\epsilon}_{t+1}$ will be imposed later.

The general solution to this class of DSGE model is given by

$$\mathbf{y}_t = \mathbf{g}\left(\mathbf{x}_t, \sigma \right) \tag{3}$$

$$\mathbf{x}_{t+1} = \mathbf{h}\left(\mathbf{x}_t, \sigma \right) + \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \tag{4}$$

$$\boldsymbol{\eta} = \begin{bmatrix} \mathbf{0} \\ \tilde{\boldsymbol{\eta}} \end{bmatrix} \tag{5}$$

where the functions $\mathbf{g}\left(\cdot, \cdot \right)$ and $\mathbf{h}\left(\cdot, \cdot \right)$ are unknown. We will therefore approximate these functions up to any desired order. This is done around the non-stochastic steady state, i.e. $\mathbf{x}_t = \mathbf{x}_{ss}$ and $\sigma = 0$. Formally, the expression for non-stochastic steady state is given as the solution of $\left(\mathbf{y}_{ss}, \mathbf{x}_{ss} \right)$ to

$$\mathbf{f}\left(\mathbf{y}_{ss}, \mathbf{y}_{ss}, \mathbf{x}_{ss}, \mathbf{x}_{ss} \right) = \mathbf{0}. \tag{6}$$

Note also that $\mathbf{x}_{ss} = \mathbf{h}\left(\mathbf{x}_{ss}, 0 \right)$ and $\mathbf{y}_{ss} = \mathbf{g}\left(\mathbf{x}_{ss}, 0 \right)$.

2 The pruning scheme:

2.1 Second order approximation

We start by partitioning the state vector using the approximated exprresion

$$\mathbf{x}_t = \mathbf{x}_t^f + \mathbf{x}_t^s,$$

where $\mathbf{x}_t^f$ denotes the first order terms and $\mathbf{x}_t^s$ denotes the second order terms.

A second-order approximation of the state equation reads (for $j = 1, 2, ..., n_x$)

$x_{t+1}(j,1) = \mathbf{h_x}(j,:)\mathbf{x}_t + \frac{1}{2}\mathbf{x}_t'\mathbf{h_{xx}}(j,:,:)\mathbf{x}_t + \frac{1}{2}h_{\sigma\sigma}(j,1)\sigma^2 + \sigma\boldsymbol{\eta}(j,:)\boldsymbol{\epsilon}_{t+1}$
$\Updownarrow$

$$x_t^f(j,1) + x_t^s(j,1) = \mathbf{h_x}(j,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) + \frac{1}{2}\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right)'\mathbf{h_{xx}}(j,:,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) + \frac{1}{2}h_{\sigma\sigma}(j,1)\sigma^2 + \sigma\boldsymbol{\eta}(j,:)\boldsymbol{\epsilon}_{t+1}$$
$\Updownarrow$

$$x_t^f(j,1) + x_t^s(j,1) = \mathbf{h_x}(j,:)\mathbf{x}_t^f + \mathbf{h_x}(j,:)\mathbf{x}_t^s + \frac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{h_{xx}}(j,:,:) + \left(\mathbf{x}_t^s\right)'\mathbf{h_{xx}}(j,:,:)\right)\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) + \frac{1}{2}h_{\sigma\sigma}(j,1)\sigma^2 + \sigma\boldsymbol{\eta}(j,:)\boldsymbol{\epsilon}_{t+1}$$
$\Updownarrow$

$$x_t^f(j,1) + x_t^s(j,1) = \mathbf{h_x}(j,:)\mathbf{x}_t^f + \mathbf{h_x}(j,:)\mathbf{x}_t^s + \frac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{h_{xx}}(j,:,:)\mathbf{x}_t^f + \left(\mathbf{x}_t^f\right)'\mathbf{h_{xx}}(j,:,:)\mathbf{x}_t^s + \left(\mathbf{x}_t^s\right)'\mathbf{h_{xx}}(j,:,:)\mathbf{x}_t^f + \left(\mathbf{x}_t^s\right)'\mathbf{h_{xx}}(j,:,:)\mathbf{x}_t^s\right) + \frac{1}{2}h_{\sigma\sigma}(j,1)\sigma^2 + \sigma\boldsymbol{\eta}(j,:)\boldsymbol{\epsilon}_{t+1}$$
$\Updownarrow$

$$x_t^f(j,1) + x_t^s(j,1) = \mathbf{h_x}(j,:)\mathbf{x}_t^f + \mathbf{h_x}(j,:)\mathbf{x}_t^s + \frac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{h_{xx}}(j,:,:)\mathbf{x}_t^f + 2\left(\mathbf{x}_t^f\right)'\mathbf{h_{xx}}(j,:,:)\mathbf{x}_t^s + \left(\mathbf{x}_t^s\right)'\mathbf{h_{xx}}(j,:,:)\mathbf{x}_t^s\right) + \frac{1}{2}h_{\sigma\sigma}(j,1)\sigma^2 + \sigma\boldsymbol{\eta}(j,:)\boldsymbol{\epsilon}_{t+1}$$
due to the symmetry of $\mathbf{h_{xx}}(j,:,:)$.

A law of motion for the first order terms is thus

$$x_t^f(j,1) = \mathbf{h_x}(j,:)\mathbf{x}_t^f + \sigma\boldsymbol{\eta}(j,:)\boldsymbol{\epsilon}_{t+1}$$

A law of motion for the second order terms is thus

$$x_t^s(j,1) = \mathbf{h_x}(j,:)\mathbf{x}_t^s + \frac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{h_{xx}}(j,:,:)\mathbf{x}_t^f\right) + \frac{1}{2}h_{\sigma\sigma}(j,1)\sigma^2$$

Inserting the decomposition of the state variables into the control variables we get (for $i = 1, 2, ...n_y$)
$y_t^s(i,1) = \mathbf{g_x}(i,:)\mathbf{x}_t + \frac{1}{2}\mathbf{x}_t'\mathbf{g_{xx}}(i,:,:)\mathbf{x}_t + \frac{1}{2}g_{\sigma\sigma}(i,1)\sigma^2$
$\Updownarrow$

$$y_t^s(i,1) = \mathbf{g_x}(i,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) + \frac{1}{2}\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right)'\mathbf{g_{xx}}(i,:,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) + \frac{1}{2}g_{\sigma\sigma}(i,1)\sigma^2$$
$\Updownarrow$

$$y_t^s(i,1) = \mathbf{g_x}(i,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) + \frac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{g_{xx}}(i,:,:) + \left(\mathbf{x}_t^s\right)'\mathbf{g_{xx}}(i,:,:)\right)\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) + \frac{1}{2}g_{\sigma\sigma}(i,1)\sigma^2$$
$\Updownarrow$

$$y_t^s(i,1) = \mathbf{g_x}(i,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) + \frac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{g_{xx}}(i,:,:)\mathbf{x}_t^f + 2\left(\mathbf{x}_t^f\right)'\mathbf{g_{xx}}(i,:,:)\mathbf{x}_t^s + \left(\mathbf{x}_t^s\right)'\mathbf{g_{xx}}(i,:,:)\mathbf{x}_t^s\right) + \frac{1}{2}g_{\sigma\sigma}(i,1)\sigma^2$$

due to the symmetry of $\mathbf{g_{xx}}(i,:,:)$

We want to preserve terms up to second order, hence the pruned approximation is

$$y_t^s(i,1) = \mathbf{g_x}(i,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) + \frac{1}{2}\left(\mathbf{x}_t^f\right)'\mathbf{g_{xx}}(i,:,:)\,\mathbf{x}_t^f + \frac{1}{2}g_{\sigma\sigma}(i,1)\,\sigma^2$$

because $\left(\mathbf{x}_t^f\right)'\mathbf{g_{xx}}(i,:,:)\,\mathbf{x}_t^s$ is a third order term and $\left(\mathbf{x}_t^s\right)'\mathbf{g_{xx}}(i,:,:)\,\mathbf{x}_t^s$ is a fourth order term

2.2 Third order approximation

We decompose the state vector using the approximated exprresion

$$\mathbf{x}_t = \mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd},$$

where the new term $\mathbf{x}_t^{rd}$ denotes the third order term.

A third order approximation of the state equation reads (for $j = 1, 2, ..., n_x$)

$$\begin{aligned} x_{t+1}(j,1) &= \mathbf{h_x}(j,:)\,\mathbf{x}_t + \sigma\boldsymbol{\eta}(j,:)\,\boldsymbol{\epsilon}_{t+1} \\ &+ \tfrac{1}{2}\mathbf{x}_t'\mathbf{h_{xx}}(j,:,:)\,\mathbf{x}_t + \tfrac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2 \\ &+ \tfrac{1}{6}\mathbf{x}_t'\begin{bmatrix} \mathbf{x}_t'\mathbf{h_{xxx}}(j,1,:,:)\mathbf{x}_t \\ ... \\ \mathbf{x}_t'\mathbf{h_{xxx}}(j,n_x,:,:)\mathbf{x}_t \end{bmatrix} + \tfrac{3}{6}\mathbf{h_{\sigma\sigma x}}(j,:)\,\sigma^2\mathbf{x}_t + \tfrac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3 \end{aligned}$$

$\Updownarrow$

$$\begin{aligned} x_t^f(j,1) + x_t^s(j,1) + x_t^{rd}(j,1) &= \mathbf{h_x}(j,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \sigma\boldsymbol{\eta}(j,:)\,\boldsymbol{\epsilon}_{t+1} \\ &+ \tfrac{1}{2}\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right)'\mathbf{h_{xx}}(j,:,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \tfrac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2 \\ &+ \tfrac{1}{6}\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right)'\begin{bmatrix} \left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right)'\mathbf{h_{xxx}}(j,1,:,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) \\ ... \\ \left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right)'\mathbf{h_{xxx}}(j,n_x,:,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) \end{bmatrix} \\ &+ \tfrac{3}{6}\mathbf{h_{\sigma\sigma x}}(j,:)\,\sigma^2\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \tfrac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3 \end{aligned}$$

$\Updownarrow$

$$\begin{aligned} x_t^f(j,1) + x_t^s(j,1) + x_t^{rd}(j,1) &= \mathbf{h_x}(j,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \sigma\boldsymbol{\eta}(j,:)\,\boldsymbol{\epsilon}_{t+1} \\ &+ \tfrac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{h_{xx}}(j,:,:) + \left(\mathbf{x}_t^s\right)'\mathbf{h_{xx}}(j,:,:) + \left(\mathbf{x}_t^{rd}\right)'\mathbf{h_{xx}}(j,:,:)\right)\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \tfrac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2 \\ &+ \tfrac{1}{6}\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right)'\times \\ &\begin{bmatrix} \left(\left(\mathbf{x}_t^f\right)'\mathbf{h_{xxx}}(j,1,:,:) + \left(\mathbf{x}_t^s\right)'\mathbf{h_{xxx}}(j,1,:,:) + \left(\mathbf{x}_t^{rd}\right)'\mathbf{h_{xxx}}(j,1,:,:)\right)\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) \\ ... \\ \left(\left(\mathbf{x}_t^f\right)'\mathbf{h_{xxx}}(j,n_x,:,:) + \left(\mathbf{x}_t^s\right)'\mathbf{h_{xxx}}(j,n_x,:,:) + \left(\mathbf{x}_t^{rd}\right)'\mathbf{h_{xxx}}(j,n_x,:,:)\right)\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) \end{bmatrix} \\ &+ \tfrac{3}{6}\mathbf{h_{\sigma\sigma x}}(j,:)\,\sigma^2\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \tfrac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3 \end{aligned}$$

$\Updownarrow$

$$\begin{aligned} x_t^f(j,1) + x_t^s(j,1) + x_t^{rd}(j,1) &= \mathbf{h_x}(j,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \sigma\boldsymbol{\eta}(j,:)\,\boldsymbol{\epsilon}_{t+1} \\ &+ \tfrac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{h_{xx}}(j,:,:)\,\mathbf{x}_t^f + \left(\mathbf{x}_t^f\right)'\mathbf{h_{xx}}(j,:,:)\,\mathbf{x}_t^s + \left(\mathbf{x}_t^f\right)'\mathbf{h_{xx}}(j,:,:)\,\mathbf{x}_t^{rd}\right) \end{aligned}$$

$$+\tfrac{1}{2}\left(\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\mathbf{x}_t^f+\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\mathbf{x}_t^s+\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{2}\left(\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\mathbf{x}_t^f+\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\mathbf{x}_t^s+\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{2}h_{\sigma\sigma}\left(j,1\right)\sigma^2$$
$$+\tfrac{1}{6}\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)'\times$$
$$\begin{bmatrix}\left(\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,1,:,:)+\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xxx}}(j,1,:,:)+\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xxx}}(j,1,:,:)\right)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)\\ \ldots\\ \left(\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,n_x,:,:)+\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xxx}}(j,n_x,:,:)+\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xxx}}(j,n_x,:,:)\right)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)\end{bmatrix}$$
$$+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\left(j,:\right)\sigma^2\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\tfrac{1}{6}h_{\sigma\sigma\sigma}\left(j,1\right)\sigma^3$$
$$\Updownarrow$$

$$x_t^f\left(j,1\right)+x_t^s\left(j,1\right)+x_t^{rd}\left(j,1\right)=\mathbf{h}_{\mathbf{x}}\left(j,:\right)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\sigma\boldsymbol{\eta}\left(j,:\right)\boldsymbol{\epsilon}_{t+1}$$
$$+\tfrac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\mathbf{x}_t^f+2\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\mathbf{x}_t^s+2\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{2}\left(\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\mathbf{x}_t^s+2\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\mathbf{x}_t^{rd}+\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{2}h_{\sigma\sigma}\left(j,1\right)\sigma^2$$
$$+\tfrac{1}{6}\textstyle\sum_{\gamma=1}^{n_x}\left(x_t^f\left(\gamma,1\right)+x_t^s\left(\gamma,1\right)+x_t^{rd}\left(\gamma,1\right)\right)\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{6}\textstyle\sum_{\gamma=1}^{n_x}\left(x_t^f\left(\gamma,1\right)+x_t^s\left(\gamma,1\right)+x_t^{rd}\left(\gamma,1\right)\right)\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{6}\textstyle\sum_{\gamma=1}^{n_x}\left(x_t^f\left(\gamma,1\right)+x_t^s\left(\gamma,1\right)+x_t^{rd}\left(\gamma,1\right)\right)\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\left(j,:\right)\sigma^2\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\tfrac{1}{6}h_{\sigma\sigma\sigma}\left(j,1\right)\sigma^3$$

due to symmetry in $\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)$

$$\Updownarrow$$

$$x_t^f\left(j,1\right)+x_t^s\left(j,1\right)+x_t^{rd}\left(j,1\right)=\mathbf{h}_{\mathbf{x}}\left(j,:\right)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\sigma\boldsymbol{\eta}\left(j,:\right)\boldsymbol{\epsilon}_{t+1}$$
$$+\tfrac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\mathbf{x}_t^f+2\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\mathbf{x}_t^s+2\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{2}\left(\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\mathbf{x}_t^s+2\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\mathbf{x}_t^{rd}+\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{2}h_{\sigma\sigma}\left(j,1\right)\sigma^2$$
$$+\tfrac{1}{6}\textstyle\sum_{\gamma=1}^{n_x}\left(x_t^f\left(\gamma,1\right)+x_t^s\left(\gamma,1\right)+x_t^{rd}\left(\gamma,1\right)\right)$$
$$\times\left(\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^f+\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^s+\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{6}\textstyle\sum_{\gamma=1}^{n_x}\left(x_t^f\left(\gamma,1\right)+x_t^s\left(\gamma,1\right)+x_t^{rd}\left(\gamma,1\right)\right)$$
$$\times\left(\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^f+\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^s+\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{6}\textstyle\sum_{\gamma=1}^{n_x}\left(x_t^f\left(\gamma,1\right)+x_t^s\left(\gamma,1\right)+x_t^{rd}\left(\gamma,1\right)\right)$$
$$\times\left(\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^f+\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^s+\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\left(j,:\right)\sigma^2\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\tfrac{1}{6}h_{\sigma\sigma\sigma}\left(j,1\right)\sigma^3$$
$$\Updownarrow$$

$$x_t^f\left(j,1\right)+x_t^s\left(j,1\right)+x_t^{rd}\left(j,1\right)=\mathbf{h}_{\mathbf{x}}\left(j,:\right)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\sigma\boldsymbol{\eta}\left(j,:\right)\boldsymbol{\epsilon}_{t+1}$$

$+\frac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^f+2\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^s+2\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^{rd}\right)$
$+\frac{1}{2}\left((\mathbf{x}_t^s)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^s+2(\mathbf{x}_t^s)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^{rd}+\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^{rd}\right)$
$+\frac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2$
$+\frac{1}{6}\sum_{\gamma=1}^{n_x}\left(x_t^f(\gamma,1)+x_t^s(\gamma,1)+x_t^{rd}(\gamma,1)\right)$
$\times\left(\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^f+2\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^s+2\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^{rd}\right)$
$+\frac{1}{6}\sum_{\gamma=1}^{n_x}\left(x_t^f(\gamma,1)+x_t^s(\gamma,1)+x_t^{rd}(\gamma,1)\right)$
$\times\left((\mathbf{x}_t^s)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^s+2(\mathbf{x}_t^s)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^{rd}+\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^{rd}\right)$
$+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}(j,:)\,\sigma^2\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\frac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3$
due to symmetries in $\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)$

A law of motion for $x_t^f(j,1)$ is then (as before)

$$x_t^f(j,1)=\mathbf{h}_{\mathbf{x}}(j,:)\,\mathbf{x}_t^f+\sigma\boldsymbol{\eta}(j,:)\,\boldsymbol{\epsilon}_{t+1}$$

because we only keep first order terms

A law of motion for $x_t^s(j,1)$ is then (as before)

$$x_t^s(j,1)=\mathbf{h}_{\mathbf{x}}(j,:)\,\mathbf{x}_t^s+\frac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^f\right)+\frac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2$$

because we only keep second order terms.

A law of motion for $x_t^{rd}(j,1)$ is then

$$\begin{aligned}x_t^{rd}(j,1) &= \mathbf{h}_{\mathbf{x}}(j,:)\,\mathbf{x}_t^{rd}+\frac{2}{2}\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^s+\frac{1}{6}\sum_{\gamma=1}^{n_x}x_t^f(\gamma,1)\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^f\\ &\quad+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}(j,:)\,\sigma^2\mathbf{x}_t^f+\frac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3\end{aligned}$$

Note that σ^2 is in perturbation a variable and $\sigma^2\mathbf{x}_t^f$ is therefore a third order effect.

Inserting the decomposition of the state variables into the control variables we get (for $i=1,2,...n_y$)

$y_t^{rd}(i,1)=\mathbf{g}_{\mathbf{x}}(i,:)\,\mathbf{x}_t+\frac{1}{2}\mathbf{x}_t'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t+\frac{1}{2}g_{\sigma\sigma}(i,1)\,\sigma^2$
$+\frac{1}{6}\mathbf{x}_t'\begin{bmatrix}\mathbf{x}_t'\mathbf{g}_{\mathbf{xxx}}(i,1,:,:)\mathbf{x}_t\\ \ldots\\ \mathbf{x}_t'\mathbf{g}_{\mathbf{xxx}}(i,n_x,:,:)\mathbf{x}_t\end{bmatrix}+\frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}(i,:)\,\sigma^2\mathbf{x}_t+\frac{1}{6}g_{\sigma\sigma\sigma}(i,1)\,\sigma^3$

$\Updownarrow$

$y_t^{rd}(i,1)=\mathbf{g}_{\mathbf{x}}(i,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\frac{1}{2}\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\frac{1}{2}g_{\sigma\sigma}(i,1)\,\sigma^2$
$+\frac{1}{6}\sum_{\gamma=1}^{n_x}\left(x_t^f(\gamma,1)+x_t^s(\gamma,1)+x_t^{rd}(\gamma,1)\right)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xxx}}(i,\gamma,:,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)$
$+\frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}(i,:)\,\sigma^2\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\frac{1}{6}g_{\sigma\sigma\sigma}(i,1)\,\sigma^3$

$\Updownarrow$

$y_t^{rd}(i,1)=\mathbf{g}_{\mathbf{x}}(i,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\frac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)+(\mathbf{x}_t^s)'\mathbf{g}_{\mathbf{xx}}(i,:,:)+\left(\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\right)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)$
$+\frac{1}{2}g_{\sigma\sigma}(i,1)\,\sigma^2$

$$+\tfrac{1}{6}\sum_{\gamma=1}^{n_x}\left(x_t^f(\gamma,1)+x_t^s(\gamma,1)+x_t^{rd}(\gamma,1)\right)$$
$$\times\left(\left(\mathbf{x}_t^f\right)'\mathbf{g}_{\mathbf{xxx}}(i,\gamma,:,:)+\left(\mathbf{x}_t^s\right)'\mathbf{g}_{\mathbf{xxx}}(i,\gamma,:,:)+\left(\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xxx}}(i,\gamma,:,:)\right)$$
$$\times\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}(i,:)\,\sigma^2\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\tfrac{1}{6}g_{\sigma\sigma\sigma}(i,1)\,\sigma^3$$

$\Updownarrow$

$$\begin{aligned}
y_t^{rd}(i,1)=\;&\mathbf{g}_{\mathbf{x}}(i,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)\\
&+\tfrac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^f+\left(\mathbf{x}_t^s\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^f+\left(\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^f\right)\\
&+\tfrac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^s+\left(\mathbf{x}_t^s\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^s+\left(\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^s\right)\\
&+\tfrac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^{rd}+\left(\mathbf{x}_t^s\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^{rd}+\left(\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^{rd}\right)\\
&+\tfrac{1}{2}g_{\sigma\sigma}(i,1)\,\sigma^2\\
&+\tfrac{1}{6}\sum_{\gamma=1}^{n_x}\left(x_t^f(\gamma,1)+x_t^s(\gamma,1)+x_t^{rd}(\gamma,1)\right)\\
&\quad\times\left(\left(\mathbf{x}_t^f\right)'\mathbf{g}_{\mathbf{xxx}}(i,\gamma,:,:)+\left(\mathbf{x}_t^s\right)'\mathbf{g}_{\mathbf{xxx}}(i,\gamma,:,:)+\left(\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xxx}}(i,\gamma,:,:)\right)\\
&\quad\times\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)\\
&+\tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}(i,:)\,\sigma^2\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\tfrac{1}{6}g_{\sigma\sigma\sigma}(i,1)\,\sigma^3
\end{aligned}$$

$\Updownarrow$

$$\begin{aligned}
y_t^{rd}(i,1)=\;&\mathbf{g}_{\mathbf{x}}(i,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)\\
&+\tfrac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^f+2\left(\mathbf{x}_t^s\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^f+2\left(\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^f\right)\\
&+\tfrac{1}{2}\left(\left(\mathbf{x}_t^s\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^s+2\left(\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^s+\left(\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^{rd}\right)\\
&+\tfrac{1}{2}g_{\sigma\sigma}(i,1)\,\sigma^2\\
&+\tfrac{1}{6}\sum_{\gamma=1}^{n_x}\left(x_t^f(\gamma,1)+x_t^s(\gamma,1)+x_t^{rd}(\gamma,1)\right)\\
&\quad\times\left(\left(\mathbf{x}_t^f\right)'\mathbf{g}_{\mathbf{xxx}}(i,\gamma,:,:)+\left(\mathbf{x}_t^s\right)'\mathbf{g}_{\mathbf{xxx}}(i,\gamma,:,:)+\left(\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xxx}}(i,\gamma,:,:)\right)\\
&\quad\times\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)\\
&+\tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}(i,:)\,\sigma^2\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\tfrac{1}{6}g_{\sigma\sigma\sigma}(i,1)\,\sigma^3
\end{aligned}$$

We want to preserve terms up to third order, hence the pruned approximation is

$$\begin{aligned}
y_t^{rd}(i,1) &= \mathbf{g_x}(i,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right) \\
&+\frac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{g_{xx}}(i,:,:)\,\mathbf{x}_t^f+2\left(\mathbf{x}_t^s\right)'\mathbf{g_{xx}}(i,:,:)\,\mathbf{x}_t^f\right) \\
&+\frac{1}{2}g_{\sigma\sigma}(i,1)\,\sigma^2 \\
&+\frac{1}{6}\sum_{\gamma=1}^{n_x}x_t^f(\gamma,1)\left(\left(\mathbf{x}_t^f\right)'\mathbf{g_{xxx}}(i,\gamma,:,:)\mathbf{x}_t^f\right) \\
&+\frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}(i,:)\,\sigma^2\mathbf{x}_t^f \\
&+\frac{1}{6}g_{\sigma\sigma\sigma}(i,1)\,\sigma^3
\end{aligned}$$

$i=1,2,...n_y$

2.3 Summary: NO pruning up to third order

The approximation of the state variables ($\mathbf{x}_t$) is here

$$\begin{aligned}
x_{t+1}(j,1) &= \mathbf{h_x}(j,:)\,\mathbf{x}_t+\frac{1}{2}\mathbf{x}_t'\mathbf{h_{xx}}(j,:,:)\,\mathbf{x}_t && (7) \\
&+\frac{1}{6}\mathbf{x}_t'\begin{bmatrix}\mathbf{x}_t'\mathbf{h_{xxx}}(j,1,:,:)\mathbf{x}_t \\ ... \\ \mathbf{x}_t'\mathbf{h_{xxx}}(j,n_x,:,:)\mathbf{x}_t\end{bmatrix} \\
&+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}(j,:)\,\sigma^2\mathbf{x}_t+\frac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2 \\
&+\frac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3+\sigma\boldsymbol{\eta}(j,:)\,\boldsymbol{\epsilon}_{t+1} && (8)
\end{aligned}$$

for $j=1,2,...,n_x$.

The approximation of the control variables ($\mathbf{y}_t$) is

$$\begin{aligned}
y_t(i,1) &= \mathbf{g_x}(i,:)\,\mathbf{x}_t && (9) \\
&+\frac{1}{2}\mathbf{x}_t'\mathbf{g_{xx}}(i,:,:)\,\mathbf{x}_t \\
&+\frac{1}{6}\mathbf{x}_t'\begin{bmatrix}\mathbf{x}_t'\mathbf{g_{xxx}}(i,1,:,:)\mathbf{x}_t \\ ... \\ \mathbf{x}_t'\mathbf{g_{xxx}}(i,n_x,:,:)\mathbf{x}_t\end{bmatrix} \\
&+\frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}(i,:)\,\sigma^2\mathbf{x}_t+\frac{1}{2}g_{\sigma\sigma}(i,1)\,\sigma^2 \\
&+\frac{1}{6}g_{\sigma\sigma\sigma}(i,1)\,\sigma^3 && (10)
\end{aligned}$$

$i=1,2,...n_y$

2.4 Summary: pruning up to third order

The approximation of the state variables is

$$\mathbf{x}_{t+1}^{f} = \mathbf{h_x}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \tag{11}$$

$$x_{t+1}^{s}(j,1) = \mathbf{h_x}(j,:)\mathbf{x}_t^s + \frac{1}{2}\left(\mathbf{x}_t^f\right)'\mathbf{h_{xx}}(j,:,:)\left(\mathbf{x}_t^f\right) + \frac{1}{2}h_{\sigma\sigma}(j,1)\sigma^2 \tag{12}$$

$$\begin{aligned} x_{t+1}^{rd}(j,1) &= \mathbf{h_x}(j,:)\mathbf{x}_t^{rd} + \frac{2}{2}\left(\mathbf{x}_t^f\right)'\mathbf{h_{xx}}(j,:,:)\left(\mathbf{x}_t^s\right) \\ &+\frac{1}{6}\left(\mathbf{x}_t^f\right)'\begin{bmatrix} \left(\mathbf{x}_t^f\right)'\mathbf{h_{xxx}}(j,1,:,:)\left(\mathbf{x}_t^f\right) \\ ... \\ \left(\mathbf{x}_t^f\right)'\mathbf{h_{xxx}}(j,n_x,:,:)\left(\mathbf{x}_t^f\right) \end{bmatrix} \\ &+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}(j,:)\sigma^2\mathbf{x}_t^f + \frac{1}{6}h_{\sigma\sigma\sigma}(j,1)\sigma^3 \end{aligned} \tag{13}$$

$$\mathbf{x}_{t+1} = \mathbf{x}_{t+1}^f + \mathbf{x}_{t+1}^s + \mathbf{x}_{t+1}^{rd} \tag{14}$$

for $j = 1, 2, ..., n_x$.

The approximation of the control variables ($\mathbf{y}_t$) is

$$\begin{aligned} y_t^{rd}(i,1) &= \mathbf{g_x}(i,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) \\ &+\frac{1}{2}\left(\mathbf{x}_t^f\right)'\mathbf{g_{xx}}(i,:,:)\left(\mathbf{x}_t^f + 2\mathbf{x}_t^s\right) \\ &+\frac{1}{6}\left(\mathbf{x}_t^f\right)'\begin{bmatrix} \left(\mathbf{x}_t^f\right)'\mathbf{g_{xxx}}(i,1,:,:)\left(\mathbf{x}_t^f\right) \\ ... \\ \left(\mathbf{x}_t^f\right)'\mathbf{g_{xxx}}(i,n_x,:,:)\left(\mathbf{x}_t^f\right) \end{bmatrix} \\ &+\frac{1}{2}g_{\sigma\sigma}(i,1)\sigma^2 + \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}(i,:)\sigma^2\mathbf{x}_t^f \\ &+\frac{1}{6}g_{\sigma\sigma\sigma}(i,1)\sigma^3 \end{aligned} \tag{15}$$

for $i = 1, ..., n_y$.

2.5 Increasing efficiency for the simulation in FORTRAN

When simulating the pruned state space system, the efficency can be improved by re-expressing some of the sums in the matrices and by using some of the symmetry in the second and third order terms (due to Young's theorem). This is useful in FORTRAN because we can reduce the number of summations. However, in MATLAB, this trick does not work as it induces more loops.

First, to re-express some of the summations implied by the matrix notation, recall the following rules for the vec and kronecker operators:

1. $vec(\mathbf{A}+\mathbf{B}) = vec(\mathbf{A}) + vec(\mathbf{B})$
2. $\mathbf{A}\otimes\mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & ... & a_{1n_x}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & ... & a_{2n_x}\mathbf{B} \\ ... & ... & ... & ... \\ a_{n_x1}\mathbf{B} & a_{n_x2}\mathbf{B} & ... & a_{n_xn_x}\mathbf{B} \end{bmatrix}$

3. $vec\,(\mathbf{ABC}) = (\mathbf{C}' \otimes \mathbf{A})\,vec\,(\mathbf{B})$ hence $\mathbf{x}_t'\mathbf{A}\mathbf{x}_t = vec\,(\mathbf{x}_t'\mathbf{A}\mathbf{x}_t) = (\mathbf{x}_t' \otimes \mathbf{x}_t')\,vec\,(\mathbf{A})$

4. $(\mathbf{A} \otimes \mathbf{B})' = (\mathbf{A}' \otimes \mathbf{B}')$ and hence $vec\,(\mathbf{A} \otimes \mathbf{B})' = vec\,(\mathbf{A}' \otimes \mathbf{B}')$

5. $(\mathbf{A} \otimes \mathbf{B})\,(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}$ if $\mathbf{AC}$ and $\mathbf{BD}$ are defined

6. $(\mathbf{A} + \mathbf{B}) \otimes (\mathbf{C} + \mathbf{D}) = \mathbf{A} \otimes \mathbf{C} + \mathbf{A} \otimes \mathbf{D} + \mathbf{B} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{D}$ if $\mathbf{A} + \mathbf{B}$ and $\mathbf{C} + \mathbf{D}$ are defined

7. $[\mathbf{x}_t' \otimes \mathbf{x}_t'] = vec\,([\mathbf{x}_t\mathbf{x}_t'])' \iff [\mathbf{x}_t \otimes \mathbf{x}_t] = vec\,([\mathbf{x}_t\mathbf{x}_t'])$

where $\mathbf{x}_t$ has dimension $n_x \times 1$ and $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ have dimension $n_x \times n_x$. Hence, we may also write the terms of the form $\mathbf{x}_t'\mathbf{h}_{\mathbf{xx}}\,(j,:,:)\,\mathbf{x}_t$ in the following way

$$\begin{aligned}
\mathbf{x}_t'\mathbf{h}_{\mathbf{xx}}\,(j,:,:)\,\mathbf{x}_t &= (\mathbf{x}_t' \otimes \mathbf{x}_t')\,vec\,(\mathbf{h}_{\mathbf{xx}}\,(j,:,:)) \\
&= vec\,(\mathbf{h}_{\mathbf{xx}}\,(j,:,:))'\,(\mathbf{x}_t \otimes \mathbf{x}_t) \\
&= vec\,(\mathbf{h}_{\mathbf{xx}}\,(j,:,:))'\,vec\,([\mathbf{x}_t\mathbf{x}_t'])
\end{aligned}$$

To exploit the symmetry in the second and third order terms, we use the *vech*-operator which stacks all elements of a matrix on or below the diagonal. For instance, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $vech\,(A) = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{22} \\ a_{32} \\ a_{33} \end{bmatrix}$.

It then holds that

$$vec\,(\mathbf{h}_{\mathbf{xx}}\,(j,:,:))'\,vec\,([\mathbf{x}_t\mathbf{x}_t']) = vech\,(2\mathbf{h}_{\mathbf{xx}}\,(j,:,:) - diag\,(\mathbf{h}_{\mathbf{xx}}\,(j,:,:)))'\,vech\,(\mathbf{x}_t\mathbf{x}_t') \tag{16}$$

Here, $diag\,(\mathbf{h}_{\mathbf{xx}}\,(j,:,:))$ is an $n_x \times n_x$ with zeros except at the diagonal where the matrix has the diagonal elements of $\mathbf{h}_{\mathbf{xx}}\,(j,:,:)$ for $i = 1, ..., n_x$. To realize the validity of the expression in (16), consider

$$\begin{aligned}
&vec\,(\mathbf{h}_{\mathbf{xx}}\,(j,:,:))'\,vec\,([\mathbf{x}_t\mathbf{x}_t']) \\
&= \mathbf{x}_t'\mathbf{h}_{\mathbf{xx}}\,(j,:,:)\,\mathbf{x}_t \\
&= \textstyle\sum_{h=1}^{n_x}\sum_{k=1}^{n_x}\mathbf{h}_{\mathbf{xx}}\,(j,:,:)\,x_t\,(h)\,x_t\,(k) \\
&= \textstyle\sum_{h=1}^{n_x}\mathbf{h}_{\mathbf{xx}}\,(j,h,h)\,x_t\,(h)^2 + 2\sum_{h=1}^{n_x}\sum_{k=h+1}^{n_x}\mathbf{h}_{\mathbf{xx}}\,(j,:,:)\,x_t\,(h)\,x_t\,(k) \\
&= \mathbf{x}_t'diag(\mathbf{h}_{\mathbf{xx}}\,(j,:,:))\mathbf{x}_t + 2\textstyle\sum_{h=1}^{n_x}\sum_{k=h+1}^{n_x}\mathbf{h}_{\mathbf{xx}}\,(j,:,:)\,x_t\,(h)\,x_t\,(k) \\
&= \mathbf{x}_t'diag(\mathbf{h}_{\mathbf{xx}}\,(j,:,:))\mathbf{x}_t + 2\left[vech\,(\mathbf{h}_{\mathbf{xx}}\,(j,:,:))'\,vech\,(\mathbf{x}_t\mathbf{x}_t') - \mathbf{x}_t'diag(\mathbf{h}_{\mathbf{xx}}\,(j,:,:))\mathbf{x}_t\right] \\
&= 2vech\,(\mathbf{h}_{\mathbf{xx}}\,(j,:,:))'\,vech\,(\mathbf{x}_t\mathbf{x}_t') - \mathbf{x}_t'diag(\mathbf{h}_{\mathbf{xx}}\,(j,:,:))\mathbf{x}_t \\
&= 2vech\,(\mathbf{h}_{\mathbf{xx}}\,(j,:,:))'\,vech\,(\mathbf{x}_t\mathbf{x}_t') - vec\,(diag(\mathbf{h}_{\mathbf{xx}}\,(j,:,:)))'\,vec\,(\mathbf{x}_t\mathbf{x}_t') \\
&= 2vech\,(\mathbf{h}_{\mathbf{xx}}\,(j,:,:))'\,vech\,(\mathbf{x}_t\mathbf{x}_t') - vech\,(diag(\mathbf{h}_{\mathbf{xx}}\,(j,:,:)))'\,vech\,(\mathbf{x}_t\mathbf{x}_t') \\
&= [2vech\,(\mathbf{h}_{\mathbf{xx}}\,(j,:,:)) - vech\,(diag(\mathbf{h}_{\mathbf{xx}}\,(j,:,:)))]'\,vech\,(\mathbf{x}_t\mathbf{x}_t')
\end{aligned}$$

WITHOUT PRUNING:

For the state variables in (7), we have

$$\begin{aligned} x_{t+1}(j,1) &= \mathbf{h_x}(j,:)\mathbf{x}_t + \hat{\mathbf{H}}_{\mathbf{xx}}(j,:)\,vech(\mathbf{x}_t\mathbf{x}_t') \\ &+\mathbf{x}_t'\begin{bmatrix} \hat{\mathbf{H}}_{\mathbf{xxx}}(j,1,:)\,vech(\mathbf{x}_t\mathbf{x}_t') \\ ... \\ \hat{\mathbf{H}}_{\mathbf{xxx}}(j,n_x,:)\,vech(\mathbf{x}_t\mathbf{x}_t') \end{bmatrix} \\ &+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}(j,:)\,\sigma^2\mathbf{x}_t + \frac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2 \\ &+\frac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3 + \sigma\boldsymbol{\eta}(j,:)\,\boldsymbol{\epsilon}_{t+1} \end{aligned} \tag{17}$$

for $j = 1, 2, ..., n_x$ where we define

$$\hat{\mathbf{H}}_{\mathbf{xx}}(1:n_x, 1:n_x(n_x+1)/2) = \frac{1}{2}\begin{bmatrix} vech(2\mathbf{h}_{\mathbf{xx}}(1,:,:) - diag(\mathbf{h}_{\mathbf{xx}}(1,:,:)))' \\ ... \\ vech(2\mathbf{h}_{\mathbf{xx}}(n_x,:,:) - diag(\mathbf{h}_{\mathbf{xx}}(n_x,:,:)))' \end{bmatrix} \tag{18}$$

$$\hat{\mathbf{H}}_{\mathbf{xxx}}(j, 1:n_x, 1:n_x(n_x+1)/2) = \frac{1}{6}\begin{bmatrix} vech(2\mathbf{h}_{\mathbf{xxx}}(j,1,:,:) - diag(\mathbf{h}_{\mathbf{xxx}}(j,1,:,:)))' \\ ... \\ vech(2\mathbf{h}_{\mathbf{xxx}}(j,n_x,:,:) - diag(\mathbf{h}_{\mathbf{xxx}}(j,n_x,:,:)))' \end{bmatrix} \tag{19}$$

for $j = 1, 2, ..., n_x$. The advantage of this formulation compared to one which use all the symmetry in the third order terms is simply that we only need to compute $vech(\mathbf{x}_t\mathbf{x}_t')$ once.

For the control variables in (9), we have

$$\begin{aligned} y_t(i,1) &= \mathbf{g_x}(i,:)\mathbf{x}_t \\ &+\hat{\mathbf{G}}_{\mathbf{xx}}(i,:)\,vech(\mathbf{x}_t\mathbf{x}_t') \\ &+\mathbf{x}_t'\begin{bmatrix} \hat{\mathbf{G}}_{\mathbf{xxx}}(i,1,:)\,vech(\mathbf{x}_t\mathbf{x}_t') \\ ... \\ \hat{\mathbf{G}}_{\mathbf{xxx}}(i,n_x,:)\,vech(\mathbf{x}_t\mathbf{x}_t') \end{bmatrix} \\ &+\frac{1}{2}g_{\sigma\sigma}(i,1)\,\sigma^2 + \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}(i,:)\,\sigma^2\mathbf{x}_t \\ &+\frac{1}{6}g_{\sigma\sigma\sigma}(i,1)\,\sigma^3 \end{aligned} \tag{20}$$

for $i = 1, 2, ..., n_y$ where we define

$$\hat{\mathbf{G}}_{\mathbf{xx}}(1:n_y, 1:n_x(n_x+1)/2) = \frac{1}{2}\begin{bmatrix} vech(2\mathbf{g}_{\mathbf{xx}}(1,:,:) - diag(\mathbf{g}_{\mathbf{xx}}(1,:,:)))' \\ ... \\ vech(2\mathbf{g}_{\mathbf{xx}}(n_y,:,:) - diag(\mathbf{g}_{\mathbf{xx}}(n_y,:,:)))' \end{bmatrix} \tag{21}$$

$$\hat{\mathbf{G}}_{\mathbf{xxx}}(i, 1:n_x, 1:n_x(n_x+1)/2) = \frac{1}{6}\begin{bmatrix} vech(2\mathbf{g}_{\mathbf{xxx}}(i,1,:,:) - diag(\mathbf{g}_{\mathbf{xxx}}(i,1,:,:)))' \\ ... \\ vech(2\mathbf{g}_{\mathbf{xxx}}(i,n_x,:,:) - diag(\mathbf{g}_{\mathbf{xxx}}(i,n_x,:,:)))' \end{bmatrix} \tag{22}$$

for $i = 1, 2, ..., n_y$.

WITH PRUNING:

For the state variables in (12) and (13), we have

$$x_{t+1}^s(j,1) = \mathbf{h_x}\mathbf{x}_t^s + \hat{\mathbf{H}}_{\mathbf{xx}}(j,:)\,vech\left(\left(\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right) + \frac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2 \tag{23}$$

$$
\begin{aligned}
x_{t+1}^{rd}(j,1) &= \mathbf{h_x}\mathbf{x}_t^{rd} + \hat{\mathbf{H}}_{\mathbf{xx}}(j,:)\left(vech\left(\left(\mathbf{x}_t^f\right)(\mathbf{x}_t^s)'\right) + vech\left((\mathbf{x}_t^s)\left(\mathbf{x}_t^f\right)'\right)\right) \\
&+ \left(\mathbf{x}_t^f\right)' \begin{bmatrix} \hat{\mathbf{H}}_{\mathbf{xxx}}(j,1,:)\, vech\left(\left(\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right) \\ \dots \\ \hat{\mathbf{H}}_{\mathbf{xxx}}(j,n_x,:)\, vech\left(\left(\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right) \end{bmatrix} \\
&+ \frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}(j,:)\,\sigma^2\mathbf{x}_t^f + \frac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3
\end{aligned} \tag{24}
$$

For the control variables in (15), we have

$$
\begin{aligned}
y_t^{rd}(i,1) &= \mathbf{g_x}(i,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) \\
&+ \hat{\mathbf{G}}_{\mathbf{xx}}(i,:)\left[vech\left(\left(\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right) + vech\left(\left(\mathbf{x}_t^f\right)(\mathbf{x}_t^s)'\right) + vech\left((\mathbf{x}_t^s)\left(\mathbf{x}_t^f\right)'\right)\right] \\
&+ \left(\mathbf{x}_t^f\right)' \begin{bmatrix} \hat{\mathbf{G}}_{\mathbf{xxx}}(i,1,:)\, vech\left(\left(\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right) \\ \dots \\ \hat{\mathbf{G}}_{\mathbf{xxx}}(i,n_x,:)\, vech\left(\left(\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right) \end{bmatrix} \\
&+ \frac{1}{2}g_{\sigma\sigma}(i,1)\,\sigma^2 + \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}(i,:)\,\sigma^2\mathbf{x}_t^f + \frac{1}{6}g_{\sigma\sigma\sigma}(i,1)\,\sigma^3
\end{aligned} \tag{25}
$$

2.6 Increasing efficiency for the simulation in MATLAB

In MATLAB the most important thing is to avoid for-loops. We therefore provide a representation based on the kronecker product which does not require any loops. Even without using the symmetry in the non-linear terms, this greatly increases the execution speed in MATLAB. Note first that

$\mathbf{x}_t'\mathbf{h_{xx}}(j,:,:)\mathbf{x}_t = reshape(\mathbf{h_{xx}}, n_x, n_x^2)\,(\mathbf{x}_t \otimes \mathbf{x}_t)$

where

$$
reshape(\mathbf{h_{xx}}, n_x, n_x^2) = \begin{bmatrix} \mathbf{h_{xx}}(1,1:n_x,1)' & \mathbf{h_{xx}}(1,1:n_x,2)' & \dots & \mathbf{h_{xx}}(1,1:n_x,n_x)' \\ \mathbf{h_{xx}}(2,1:n_x,1)' & \mathbf{h_{xx}}(2,1:n_x,2)' & \dots & \mathbf{h_{xx}}(1,1:n_x,n_x)' \\ \dots & \dots & \dots & \dots \\ \mathbf{h_{xx}}(n_x,1:n_x,1)' & \mathbf{h_{xx}}(n_x,1:n_x,2)' & \dots & \mathbf{h_{xx}}(n_x,1:n_x,n_x)' \end{bmatrix}
$$

And for the third order terms:

$$
\mathbf{x}_t' \begin{bmatrix} \mathbf{x}_t'\mathbf{h_{xxx}}(j,1,:,:)\mathbf{x}_t \\ \dots \\ \mathbf{x}_t'\mathbf{h_{xxx}}(j,n_x,:,:)\mathbf{x}_t \end{bmatrix} = \textstyle\sum_{j_1=1}^{n_x} x_t(j_1,1)\,\mathbf{x}_t'\mathbf{h_{xxx}}(j,j_1,:,:)\mathbf{x}_t
$$

$$
= \textstyle\sum_{j_1=1}^{n_x}\sum_{j_2=1}^{n_x}\sum_{j_3=1}^{n_x} x_t(j_1,1)\,x_t(j_2,1)\,x_t(j_3,1)\,\mathbf{h_{xxx}}(j,j_1,j_2,j_3)
$$

$$
= reshape(\mathbf{h_{xxx}}, n_x, n_x^3)\,(\mathbf{x}_t \otimes \mathbf{x}_t \otimes \mathbf{x}_t)
$$

WITHOUT PRUNING:

For the state variables in (7), we have

$$\begin{aligned}\mathbf{x}_{t+1} &= \mathbf{h}_{\mathbf{x}}\mathbf{x}_t + \tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t \otimes \mathbf{x}_t\right) + \tilde{\mathbf{H}}_{\mathbf{xxx}}\left(\mathbf{x}_t \otimes \mathbf{x}_t \otimes \mathbf{x}_t\right) \\ &+ \frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3 + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\end{aligned} \tag{26}$$

where we define

$$\tilde{\mathbf{H}}_{\mathbf{xx}} \equiv \frac{1}{2} reshape\left(\mathbf{h}_{\mathbf{xx}}, n_x, n_x^2\right) \tag{27}$$

$$\tilde{\mathbf{H}}_{\mathbf{xxx}} \equiv \frac{1}{6} reshape\left(\mathbf{h}_{\mathbf{xxx}}, n_x, n_x^3\right) \tag{28}$$

For the control variables in (9), we have

$$\mathbf{y}_t = \mathbf{g}_{\mathbf{x}}\mathbf{x}_t + \tilde{\mathbf{G}}_{\mathbf{xx}}\left(\mathbf{x}_t \otimes \mathbf{x}_t\right) + \tilde{\mathbf{G}}_{\mathbf{xxx}}\left(\mathbf{x}_t \otimes \mathbf{x}_t \otimes \mathbf{x}_t\right) \tag{29}$$

$$+ \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3 \tag{30}$$

where we define

$$\tilde{\mathbf{G}}_{\mathbf{xx}} \equiv \frac{1}{2} reshape\left(\mathbf{g}_{\mathbf{xx}}, n_y, n_x^2\right) \tag{31}$$

$$\tilde{\mathbf{G}}_{\mathbf{xxx}} \equiv \frac{1}{6} reshape\left(\mathbf{g}_{\mathbf{xxx}}, n_y, n_x^3\right) \tag{32}$$

WITH PRUNING:
For the state variables in (12) and (13), we have

$$\mathbf{x}_{t+1}^s = \mathbf{h}_{\mathbf{x}}\mathbf{x}_t^s + \tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \tag{33}$$

$$\begin{aligned}\mathbf{x}_{t+1}^{rd} &= \mathbf{h}_{\mathbf{x}}\mathbf{x}_t^{rd} + 2\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \tilde{\mathbf{H}}_{\mathbf{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) \\ &+ \frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f + \frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\end{aligned} \tag{34}$$

For the control variables in (15), we have

$$\begin{aligned}\mathbf{y}_t^{rd} &= \mathbf{g}_{\mathbf{x}}\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \tilde{\mathbf{G}}_{\mathbf{xx}}\left(\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + 2\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\right) + \tilde{\mathbf{G}}_{\mathbf{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) \\ &+ \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3\end{aligned} \tag{35}$$

3 Stastical properties: Second-order approximation

3.1 Covariance-stationary

Proposition 1:

The pruned second-order approximation for $\mathbf{x}_t^f, \mathbf{x}_t^s$, and $\mathbf{y}_t^s$ is covariance-stationary if

1. the DSGE model has a unique stable equilibrium, i.e. all eigenvalue of $\mathbf{h_x}$ have modulus less than one
2. $\boldsymbol{\epsilon}_{t+1}$ has finite fourth moment

Proof

Note first that

$x_{t+1}^s(j,1) = \mathbf{h_x}(j,:)\,\mathbf{x}_t^s + \frac{1}{2}\left(\mathbf{x}_t^f\right)'\mathbf{h_{xx}}(j,:,:)\left(\mathbf{x}_t^f\right) + \frac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2$

$\Updownarrow$

$\mathbf{x}_{t+1}^s = \mathbf{h_x}\mathbf{x}_t^s + \tilde{\mathbf{H}}_{\mathbf{xx}} vec\left(\left[\left(\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right]\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$

where $\tilde{\mathbf{H}}_{\mathbf{xx}} \equiv \frac{1}{2}reshape\left(\mathbf{h_{xx}}, n_x, n_x^2\right)$

$\Updownarrow$

$\mathbf{x}_{t+1}^s = \mathbf{h_x}\mathbf{x}_t^s + \tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$

because $\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) = vec\left(\left[\left(\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right]\right)$

We now form the extended state vector

$$\mathbf{z}_t \equiv \begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{bmatrix}$$

We know the law of motion for $\mathbf{x}_t^f$ and $\mathbf{x}_t^s$, so we only need to find the law of motion for $\mathbf{x}_t^f \otimes \mathbf{x}_t^f$. Hence consider

$\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f = \left(\mathbf{h_x}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes \left(\mathbf{h_x}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)$

$= \mathbf{h_x}\mathbf{x}_t^f \otimes \mathbf{h_x}\mathbf{x}_t^f + \mathbf{h_x}\mathbf{x}_t^f \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h_x}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$

using $(\mathbf{A}+\mathbf{B}) \otimes (\mathbf{C}+\mathbf{D}) = \mathbf{A}\otimes\mathbf{C} + \mathbf{A}\otimes\mathbf{D} + \mathbf{B}\otimes\mathbf{C} + \mathbf{B}\otimes\mathbf{D}$

$= (\mathbf{h_x}\otimes\mathbf{h_x})\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)$
$+ (\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})$

using $(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D}) = \mathbf{AC}\otimes\mathbf{BD}$

$= (\mathbf{h_x}\otimes\mathbf{h_x})\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)$
$+ (\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})((\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}) - vec(\mathbf{I}_{n_e})) + (\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\,vec(\mathbf{I}_{n_e})$

Note that $E[(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})] = vec(\mathbf{I}_{n_e})$. Thus

$$\begin{bmatrix} \mathbf{x}_{t+1}^f \\ \mathbf{x}_{t+1}^s \\ \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \end{bmatrix} = \begin{bmatrix} \mathbf{h_x} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{h_x} & \tilde{\mathbf{H}}_{\mathbf{xx}} \\ \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x} & \mathbf{h_x}\otimes\mathbf{h_x} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{n_x\times 1} \\ \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\ (\sigma\boldsymbol{\eta}\otimes\sigma\boldsymbol{\eta})\,vec(\mathbf{I}_{n_e}) \end{bmatrix}$$

$$+\left[\begin{array}{cccc} \sigma\boldsymbol{\eta} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}) & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x} & \mathbf{h_x}\otimes\sigma\boldsymbol{\eta} \end{array}\right]\left[\begin{array}{c} \boldsymbol{\epsilon}_{t+1} \\ \boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right) \\ \boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f \\ \mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1} \end{array}\right]$$

$\Updownarrow$

$$\mathbf{z}_{t+1}=\mathbf{A}\mathbf{z}_t+\mathbf{c}+\mathbf{B}\boldsymbol{\xi}_{t+1} \tag{36}$$

where $Cov\left(\boldsymbol{\xi}_{t+1},\boldsymbol{\xi}_{t-s}\right)=\mathbf{0}$ for $s=1,2,3,...$ because $\boldsymbol{\epsilon}_{t+1}$ is independent across time

The absolute value of the eigenvalues in $\mathbf{h_x}$ are all strictly less then one by assumption. Accordingly, all eigenvalues of $\mathbf{A}$ are also strictly less than one. To see this note first that

$p(\lambda)=\left|\mathbf{A}-\lambda\mathbf{I}_{2n_x+n_x^2}\right|$

$$=\left|\left[\begin{array}{ccc} \mathbf{h_x}-\lambda\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{h_x}-\lambda\mathbf{I}_{n_x} & \tilde{\mathbf{H}}_{\mathbf{xx}} \\ \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x} & \mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I}_{n_x^2} \end{array}\right]\right|$$

$$=\left|\begin{array}{cc} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{array}\right|$$

where we let

$\mathbf{B}_{11}\equiv\left[\begin{array}{cc} \mathbf{h_x}-\lambda\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_x} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{h_x}-\lambda\mathbf{I}_{n_x} \end{array}\right]$ which is $2n_x\times 2n_x$

$\mathbf{B}_{12}\equiv\left[\begin{array}{c} \mathbf{0}_{n_x\times n_x^2} \\ \tilde{\mathbf{H}}_{\mathbf{xx}} \end{array}\right]$ which is $2n_x\times n_x^2$

$\mathbf{B}_{21}\equiv\left[\begin{array}{cc} \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x} \end{array}\right]$ which is $n_x^2\times 2n_x$

$\mathbf{B}_{22}\equiv\mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I}_{n_x^2}$ which is $n_x^2\times n_x^2$

$$=\left|\mathbf{B}_{11}\right|\left|\mathbf{B}_{22}\right|$$

using $\left|\begin{array}{cc} \mathbf{U} & \mathbf{C} \\ \mathbf{0} & \mathbf{Y} \end{array}\right|=\left|\mathbf{U}\right|\left|\mathbf{Y}\right|$ where $\mathbf{U}$ is $m\times m$ and $\mathbf{Y}$ is $n\times n$

$$=\left|\left[\begin{array}{cc} \mathbf{h_x}-\lambda\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_x} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{h_x}-\lambda\mathbf{I}_{n_x} \end{array}\right]\right|\left|\mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I}_{n_x^2}\right|$$

$$=\left|\mathbf{h_x}-\lambda\mathbf{I}_{n_x}\right|\left|\mathbf{h_x}-\lambda\mathbf{I}_{n_x}\right|\left|\mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I}_{n_x^2}\right|$$

Hence, the eigenvalue λ solves the problem

$p(\lambda)=0$

$\Updownarrow$

$\left|\mathbf{h_x}-\lambda\mathbf{I}_{n_x}\right|\left|\mathbf{h_x}-\lambda\mathbf{I}_{n_x}\right|\left|\mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I}_{n_x^2}\right|=0$

$\Updownarrow$

$\left|\mathbf{h_x}-\lambda\mathbf{I}_{n_x}\right|=0$ or $\left|\mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I}_{n_x^2}\right|=0$

The absolute value of all eigenvalues to the first problem are strictly less than one. That is $|\lambda_i|<1$ $i=1,2,...,n_x$. This is also the case for the second problem because the eigenvalues to $\mathbf{h_x}\otimes\mathbf{h_x}$ are $\lambda_i\lambda_j$ for $i=1,2,...,n_x$ and $j=1,2,...,n_x$

Thus, the system in (36) is covariance stationary if $\boldsymbol{\xi}_{t+1}$ has finite first and second moment. It follows directly that $E\left[\boldsymbol{\xi}_{t+1}\right]=\mathbf{0}$ and $\boldsymbol{\xi}_{t+1}$ has finite second moments if $\boldsymbol{\epsilon}_{t+1}$ has a finite fourth moment. The latter holds by assumption.

For the control variables we have

$y_t^s(i,1) = \mathbf{g_x}(i,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) + \frac{1}{2}\left(\mathbf{x}_t^f\right)' \mathbf{g_{xx}}(i,:,:)\,\mathbf{x}_t^f + \frac{1}{2} g_{\sigma\sigma}(i,1)\,\sigma^2$
$\Updownarrow$

$\mathbf{y}_t^s = \mathbf{g_x}\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) + \tilde{\mathbf{G}}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$
where $\tilde{\mathbf{G}}_{\mathbf{xx}} \equiv \frac{1}{2} reshape\left(\mathbf{g_{xx}}, n_y, n_x^2\right)$
$\Updownarrow$

$\mathbf{y}_t^s = \begin{bmatrix} \mathbf{g_x} & \mathbf{g_x} & \tilde{\mathbf{G}}_{\mathbf{xx}} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{bmatrix} + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$
$\Updownarrow$

$\mathbf{y}_t^s = \mathbf{D}\mathbf{z}_t + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$
where $\mathbf{D} \equiv \begin{bmatrix} \mathbf{g_x} & \mathbf{g_x} & \tilde{\mathbf{G}}_{\mathbf{xx}} \end{bmatrix}$
That is $\mathbf{y}_t$ is linear function of $\mathbf{z}_t$ and $\mathbf{y}_t$ is therefore also covariance-stationary.

Q.E.D.

3.2 Method 1: Formulas for the first and second moments

This section computes first and second moments using the representation of the second-order system stated above. This method is fairly direct but has the computational disadvantage of requiring a lot of memory because we work directly with the big $\mathbf{B}$ matrix.

The system

$$\mathbf{z}_{t+1} = \mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}$$

$$\mathbf{y}_t^s = \mathbf{D}\mathbf{z}_t + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$$

The mean values are

$$E[\mathbf{z}_t] = \left(\mathbf{I}_{2n_x+n_x^2} - \mathbf{A}\right)^{-1}\mathbf{c}.$$

$$E[\mathbf{y}_t] = \mathbf{D}E[\mathbf{z}_t] + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$$

For the variances we first have that

$E\left[\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right] = E\left[\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}\right)\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}\right)'\right]$

$= E\left[\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}\right)\left(\mathbf{c}' + \mathbf{z}_t'\mathbf{A}' + \boldsymbol{\xi}_{t+1}'\mathbf{B}'\right)\right]$

$= E\left[\mathbf{c}\left(\mathbf{c}' + \mathbf{z}_t'\mathbf{A}' + \boldsymbol{\xi}_{t+1}'\mathbf{B}'\right)\right]$
$+E\left[\mathbf{A}\mathbf{z}_t\left(\mathbf{c}' + \mathbf{z}_t'\mathbf{A}' + \boldsymbol{\xi}_{t+1}'\mathbf{B}'\right)\right]$
$+E\left[\mathbf{B}\boldsymbol{\xi}_{t+1}\left(\mathbf{c}' + \mathbf{z}_t'\mathbf{A}' + \boldsymbol{\xi}_{t+1}'\mathbf{B}'\right)\right]$

$= E\left[\mathbf{c}\mathbf{c}' + \mathbf{c}\mathbf{z}_t'\mathbf{A}' + \mathbf{c}\boldsymbol{\xi}_{t+1}'\mathbf{B}'\right]$
$+E\left[\mathbf{A}\mathbf{z}_t\mathbf{c}' + \mathbf{A}\mathbf{z}_t\mathbf{z}_t'\mathbf{A}' + \mathbf{A}\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\mathbf{B}'\right]$
$+E\left[\mathbf{B}\boldsymbol{\xi}_{t+1}\mathbf{c}' + \mathbf{B}\boldsymbol{\xi}_{t+1}\mathbf{z}_t'\mathbf{A}' + \mathbf{B}\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\mathbf{B}'\right]$

$= \mathbf{c}\mathbf{c}' + \mathbf{c}E[\mathbf{z}_t']\mathbf{A}'$
$+\mathbf{A}E[\mathbf{z}_t]\mathbf{c}' + \mathbf{A}E[\mathbf{z}_t\mathbf{z}_t']\mathbf{A}' + \mathbf{A}E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$
$+\mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$

We then note that

$$E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\right] = E\left[\left[\begin{array}{c}\mathbf{x}_t^f\\ \mathbf{x}_t^s\\ \mathbf{x}_t^f\otimes\mathbf{x}_t^f\end{array}\right]\left[\begin{array}{cccc}\boldsymbol{\epsilon}_{t+1}' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{ne}\right)\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\end{array}\right]\right]$$

$$= E\left[\begin{array}{cccc}\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}' & \mathbf{x}_t^f\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{ne}\right)\right)' & \mathbf{x}_t^f\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\\ \mathbf{x}_t^s\boldsymbol{\epsilon}_{t+1}' & \mathbf{x}_t^s\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{ne}\right)\right)' & \mathbf{x}_t^s\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\\ \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\boldsymbol{\epsilon}_{t+1}' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{ne}\right)\right)' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\end{array}\right]$$

$$= \left[\begin{array}{cccc}\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{array}\right]$$

Thus

$$E\left[\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right] = \mathbf{c}\mathbf{c}' + \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{A}E\left[\mathbf{z}_t\right]\mathbf{c}' + \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$

$$= \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' + \left(\mathbf{c}+\mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' + \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$

Note also that

$$E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t\right]' = \left(\mathbf{c}+\mathbf{A}E\left[\mathbf{z}_t\right]\right)\left(\mathbf{c}+\mathbf{A}E\left[\mathbf{z}_t\right]\right)'$$

$$= \left(\mathbf{c}+\mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' + \left(\mathbf{c}+\mathbf{A}E\left[\mathbf{z}_t\right]\right)E\left[\mathbf{z}_t'\right]\mathbf{A}'$$

$$= \left(\mathbf{c}+\mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' + \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{A}E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\mathbf{A}'$$

So

$$E\left[\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right] - E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t\right]' = \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' + \left(\mathbf{c}+\mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' + \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}' - \left(\mathbf{c}+\mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' - \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' - \mathbf{A}E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\mathbf{A}'$$

$$= \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}' - \mathbf{A}E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\mathbf{A}'$$

$$= \mathbf{A}\left(E\left[\mathbf{z}_t\mathbf{z}_t'\right] - E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\right)\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$

$\Updownarrow$

$$Var\left(\mathbf{z}_{t+1}\right) = \mathbf{A}Var\left(\mathbf{z}_t\right)\mathbf{A}' + \mathbf{B}Var\left(\boldsymbol{\xi}_{t+1}\right)\mathbf{B}'$$

$\Updownarrow$

$$vec\left(Var\left(\mathbf{z}_{t+1}\right)\right) = vec\left(\mathbf{A}Var\left(\mathbf{z}_t\right)\mathbf{A}'\right) + vec\left(\mathbf{B}Var\left(\boldsymbol{\xi}_{t+1}\right)\mathbf{B}'\right)$$

$\Updownarrow$

$$vec\left(Var\left(\mathbf{z}_{t+1}\right)\right) = \left(\mathbf{A}\otimes\mathbf{A}\right)vec\left(Var\left(\mathbf{z}_t\right)\right) + vec\left(\mathbf{B}Var\left(\boldsymbol{\xi}_{t+1}\right)\mathbf{B}'\right)$$

$\Updownarrow$

$$vec\left(Var\left(\mathbf{z}_{t+1}\right)\right)\left(\mathbf{I}_{(2n_x+n_x^2)^2} - \left(\mathbf{A}\otimes\mathbf{A}\right)\right) = vec\left(\mathbf{B}Var\left(\boldsymbol{\xi}_{t+1}\right)\mathbf{B}'\right)$$

$\Updownarrow$

$$vec\left(Var\left(\mathbf{z}_{t+1}\right)\right) = \left(\mathbf{I}_{(2n_x+n_x^2)^2} - \left(\mathbf{A}\otimes\mathbf{A}\right)\right)^{-1} vec\left(\mathbf{B}Var\left(\boldsymbol{\xi}_{t+1}\right)\mathbf{B}'\right)$$

Hence we only need to compute $Var\left(\boldsymbol{\xi}_{t+1}\right)$.

$$Var\left(\boldsymbol{\xi}_{t+1}\right) = E\left[\left[\begin{array}{c}\boldsymbol{\epsilon}_{t+1}\\ \boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\\ \boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\\ \mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\end{array}\right]\left[\begin{array}{c}\boldsymbol{\epsilon}_{t+1}\\ \boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\\ \boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\\ \mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\end{array}\right]'\right]$$

$$= E\left[\left[\begin{array}{c}\boldsymbol{\epsilon}_{t+1}\\ \boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\\ \boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\\ \mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\end{array}\right]\begin{array}{cccc}\boldsymbol{\epsilon}'_{t+1} & \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\end{array}\right]$$

$$= E\left[\begin{array}{cccc}\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1} & \boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)' & \boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\\ \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)\boldsymbol{\epsilon}'_{t+1} & \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\\ \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\boldsymbol{\epsilon}'_{t+1} & \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\\ \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}'_{t+1} & \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)' & \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\end{array}\right]$$

$$= \left[\begin{array}{cccc}\mathbf{I}_{n_e} & E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right] & \mathbf{0} & \mathbf{0}\\ E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}'_{t+1}\right] & E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)'\right] & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right] & E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]\\ \mathbf{0} & \mathbf{0} & E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right] & E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]\end{array}\right]$$

All elements in this matrix can be computed (and coded) directly as shown below. The variance of the control variables is then given by

$$Var\left[\mathbf{y}_t^s\right] = \mathbf{D}Var\left[\mathbf{z}_t\right]\mathbf{D}'$$

3.2.1 Computing the variance of the innovations

1) for $E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right] = E\left[\left\{\epsilon_{t+1}\left(\phi_1,1\right)\right\}_{\phi_1=1}^{n_e}\left(\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{\epsilon_{t+1}\left(\phi_3,1\right)\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Hence the quasi MATLAB codes are :

$E\_eps\_eps2 = zeros(ne,(ne)^2)$
$for\ phi1 = 1 : ne$
  $index2 = 0$
  $for\ phi2 = 1 : ne$

$for\ phi3 = 1 : ne$
$index2 = index2 + 1$
$if\ (phi1 = phi2 = phi3)$
$E\_eps\_eps2(phi1, index2) = m^3\left(\epsilon_{t+1}\left(phi1\right)\right)$
end
end
end
end

Note also that $E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right) \boldsymbol{\epsilon}_{t+1}'\right] = \left(E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]\right)'$

2) $E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)'\right]$
Here
$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)'\right]$

$$= E\left[\left(\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right) - vec\left(\mathbf{I}_{n_e}\right)\right)\left(\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)' - vec\left(\mathbf{I}_{n_e}\right)'\right)\right]$$

$$= E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)' - vec\left(\mathbf{I}_{n_e}\right)'\right)\right]$$
$$+E\left[-vec\left(\mathbf{I}_{n_e}\right)\left(\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)' - vec\left(\mathbf{I}_{n_e}\right)'\right)\right]$$

$$= E\left[\left(\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)' - \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right) vec\left(\mathbf{I}_{n_e}\right)'\right)\right]$$
$$+E\left[-vec\left(\mathbf{I}_{n_e}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)' + vec\left(\mathbf{I}_{n_e}\right) vec\left(\mathbf{I}_{n_e}\right)'\right]$$

$$= E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right] - vec\left(\mathbf{I}_{n_e}\right) vec\left(\mathbf{I}_{n_e}\right)'$$
$$-vec\left(\mathbf{I}_{n_e}\right) vec\left(\mathbf{I}_{n_e}\right)' + vec\left(\mathbf{I}_{n_e}\right) vec\left(\mathbf{I}_{n_e}\right)'$$

$$= E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right] - vec\left(\mathbf{I}_{n_e}\right) vec\left(\mathbf{I}_{n_e}\right)'$$

Here
$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$= E\left[\left\{\epsilon_{t+1}\left(\phi_1, 1\right)\left\{\epsilon_{t+1}\left(\phi_2, 1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\left(\left\{\epsilon_{t+1}\left(\phi_3, 1\right)\left\{\epsilon_{t+1}\left(\phi_4, 1\right)\right\}_{\phi_4=1}^{n_e}\right\}_{\phi_3=1}^{n_e}\right)'\right]$$

Hence the quasi MATLAB codes are
$E\_eps2\_eps2 = zeros(n_e^2, n_e^2)$
$index1 = 0$
$for\ phi1 = 1 : n_e$
$for\ phi2 = 1 : n_e$
$index1 = index1 + 1$
$index2 = 0$
$for\ phi3 = 1 : n_e$
$for\ phi4 = 1 : n_e$
$index2 = index2 + 1$
% second moments
$if\ (phi1 == phi2\ \&\&\ phi3 == phi4\ \&\&\ phi1\tilde{}= phi4)$
$E\_eps2\_eps2(index1, index2) = 1$
$elseif\ (phi1 == phi3\ \&\&\ phi2 == phi4\ \&\&\ phi1\tilde{}= phi2)$
$E\_eps2\_eps2(index1, index2) = 1$
$elseif\ (phi1 == phi4\ \&\&\ phi2 == phi3\ \&\&\ phi1\tilde{}= phi2)$
$E\_eps2\_eps2(index1, index2) = 1$

$\%$ fourth moments
$elseif(phi1 == phi2\ \&\&\ phi1 == phi3\ \&\&\ phi1 == phi4)$
$E\_eps2\_eps2(index1, index2) = m^4\left(\epsilon_{t+1}\left(phi1\right)\right)$
end
end
end
end
end

3) $E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$
Here
$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$

$$= E\left[\left\{\epsilon_{t+1}\left(\phi_1, 1\right)\left\{x_t^f\left(\gamma_1, 1\right)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\left\{\epsilon_{t+1}\left(\phi_2, 1\right)\left\{x_t^f\left(\gamma_2, 1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right]$$

Hence the quasi MATLAB codes are
$E\_epsxf\_epsxf = zeros(n_e n_x, n_x n_e)$
$index1 = 0$
$for\ phi1 = 1 : ne$
$for\ gama1 = 1 : nx$
$index1 = index1 + 1$
$index2 = 0$
$for\ phi2 = 1 : ne$
$for\ gama2 = 1 : nx$
$index2 = index2 + 1$
$if\ phi1 = phi2$
$E\_epsxf\_epsxf\left(index1, index2\right) = E\_xf\_xf(gama1, gama2)$
end
end
end
end
end
where $E\_xf\_xf = reshape(E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right], nx, nx)$

4) $E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$
Here
$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$= E\left[\left\{\epsilon_{t+1}\left(\phi_1, 1\right)\left\{x_t^f\left(\gamma_1, 1\right)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\left\{x_t^f\left(\gamma_2, 1\right)\left\{\epsilon_{t+1}\left(\phi_2, 1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right]$$

Hence the quasi MATLAB codes are
$E\_epsxf\_xfeps = zeros(n_e n_x, n_e n_x)$
$index1 = 0$
$for\ phi1 = 1 : ne$
$for\ gama1 = 1 : nx$
$index1 = index1 + 1$
$index2 = 0$

$for\ gama2 = 1 : nx$
$for\ phi2 = 1 : ne$
$index2 = index2 + 1$
$if\ phi1 = phi2$
$E\_epsxf\_xfeps\,(index1, index2) = E\_xf\_xf(gama1, gama2)$
end
end
end
end
end

5) $E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$
Here
$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right] = \left[E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]\right]'$

so $E\_xfeps\_epsxf = E\_epsxf\_xfeps'$

6) $E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$
Here
$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}'_{t+1} \otimes \left(\mathbf{x}_t^f\right)'\right)\right]$

$$= E\left[\left\{x_t^f(\gamma_1, 1)\left\{\epsilon_{t+1}(\phi_1, 1)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\left(\left\{\epsilon_{t+1}(\phi_2, 1)\left\{x_t^f(\gamma_2, 1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are
$E\_xfeps\_epsxf = zeros(n_x n_e, n_x n_e)$
$index1 = 0$
$for\ gama1 = 1 : nx$
$for\ phi1 = 1 : ne$
$index1 = index1 + 1$
$index2 = 0$
$for\ phi2 = 1 : ne$
$for\ gama2 = 1 : nx$
$index2 = index2 + 1$
$if\ phi1 = phi2$
$E\_xfeps\_epsxf\,(index1, index2) = E\_xf\_xf(gama1, gama2)$
end
end
end
end
end
where $E\_xf\_xf = reshape(E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right], nx, nx)$

3.3 Method 2: Formulas for the first and second moments

This section computes first and second moments using a slightly different representation of the second-order system than stated above. (Basically, this was the first representation we considered for computing these moments). The advantage of this method is that it compared to Method 1 is less memory intensive because some of the matrix multiplications are done by hand.

We start by deriving an alternative representation of the pruned state space system (the old representation). Hence consider

$$\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f = \left(\mathbf{h_x}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes \left(\mathbf{h_x}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)$$

$$= \mathbf{h_x}\mathbf{x}_t^f \otimes \mathbf{h_x}\mathbf{x}_t^f + \mathbf{h_x}\mathbf{x}_t^f \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h_x}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$

using $(\mathbf{A}+\mathbf{B}) \otimes (\mathbf{C}+\mathbf{D}) = \mathbf{A}\otimes\mathbf{C} + \mathbf{A}\otimes\mathbf{D} + \mathbf{B}\otimes\mathbf{C} + \mathbf{B}\otimes\mathbf{D}$

$$= (\mathbf{h_x} \otimes \mathbf{h_x})\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) + (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})$$

using $(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D}) = \mathbf{AC}\otimes\mathbf{BD}$

$$= (\mathbf{h_x} \otimes \mathbf{h_x})\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \mathbf{v}(t+1)$$

where

$$\mathbf{v}(t+1) = (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) + (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})$$

Note that $E[\mathbf{v}(t+1)] = (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\,vec(\mathbf{I}_{n_e})$ because $\boldsymbol{\epsilon}_{t+1}$ is independent across time and therefore also independent of $\mathbf{x}_t^f$. Moreover, $E\left[\mathbf{x}_t^f\right] = 0$ and $E[\boldsymbol{\epsilon}_{t+1}] = 0$.

Thus

$$\begin{bmatrix} \mathbf{x}_{t+1}^f \\ \mathbf{x}_{t+1}^s \\ \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \end{bmatrix} = \begin{bmatrix} \mathbf{h_x} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_x^2} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{h_x} & \tilde{\mathbf{H}}_{\mathbf{xx}} \\ \mathbf{0}_{n_x^2 \times n_x} & \mathbf{0}_{n_x^2 \times n_x} & \mathbf{h_x} \otimes \mathbf{h_x} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{n_x \times 1} \\ \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\ (\sigma\boldsymbol{\eta} \otimes \sigma\boldsymbol{\eta})\,vec(\mathbf{I}_{n_e}) \end{bmatrix} + \begin{bmatrix} \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \\ \mathbf{0}_{n_x \times 1} \\ \mathbf{v}(t+1) - (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\,vec(\mathbf{I}_{n_e}) \end{bmatrix}$$

$\Updownarrow$

$$\mathbf{z}_{t+1} = \mathbf{c} + \mathbf{A}\mathbf{z}_t + \tilde{\boldsymbol{\xi}}_{t+1} \tag{37}$$

where $Cov\left(\tilde{\boldsymbol{\xi}}_{t+1}, \tilde{\boldsymbol{\xi}}_{t-s}\right) = \mathbf{0}$ for $s = 1, 2, 3, ...$ because $\boldsymbol{\epsilon}_{t+1}$ is independent across time. The expression for the controls are as above, i.e.

$$\mathbf{y}_t^s = \mathbf{D}\mathbf{z}_t + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$$

The mean values are

$$E[\mathbf{z}_t] = \left(\mathbf{I}_{2n_x+n_x^2} - \mathbf{A}\right)^{-1}\mathbf{c}$$

$$E[\mathbf{y}_t] = \mathbf{D}E[\mathbf{z}_t] + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$$

and the covariance matrix is

$$Var(\mathbf{z}_{t+1}) = \mathbf{A}Var(\mathbf{z}_t)\mathbf{A}' + Var\left(\tilde{\boldsymbol{\xi}}_{t+1}\right)$$

$\Updownarrow$

$$vec\left(Var\left(\mathbf{z}_{t+1}\right)\right)=vec\left(\mathbf{A}Var\left(\mathbf{z}_{t}\right)\mathbf{A}'\right)+vec\left(Var\left(\tilde{\boldsymbol{\xi}}_{t+1}\right)\right)$$
$$\Updownarrow$$
$$vec\left(Var\left(\mathbf{z}_{t+1}\right)\right)=\left(\mathbf{A}\otimes\mathbf{A}\right)vec\left(Var\left(\mathbf{z}_{t}\right)\right)+vec\left(Var\left(\tilde{\boldsymbol{\xi}}_{t+1}\right)\right)$$
$$\Updownarrow$$
$$vec\left(Var\left(\mathbf{z}_{t+1}\right)\right)\left(\mathbf{I}_{(2n_x+n_x^2)^2}-\left(\mathbf{A}\otimes\mathbf{A}\right)\right)=vec\left(Var\left(\tilde{\boldsymbol{\xi}}_{t+1}\right)\right)$$
$$\Updownarrow$$
$$vec\left(Var\left(\mathbf{z}_{t+1}\right)\right)=\left(\mathbf{I}_{(2n_x+n_x^2)^2}-\left(\mathbf{A}\otimes\mathbf{A}\right)\right)^{-1}vec\left(Var\left(\tilde{\boldsymbol{\xi}}_{t+1}\right)\right)$$

The variance of the control variables is then given by

$$Var\left[\mathbf{y}_t^s\right]=\mathbf{D}Var\left[\mathbf{z}_t\right]\mathbf{D}'$$

Hence we only need to compute $Var\left(\tilde{\boldsymbol{\xi}}_{t+1}\right)$.

$$Var\left(\tilde{\boldsymbol{\xi}}_{t+1}\right)=E\left(\begin{bmatrix}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\\ \mathbf{0}_{n_x\times 1}\\ \mathbf{v}\left(t+1\right)-\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right)\end{bmatrix}\begin{bmatrix}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\\ \mathbf{0}_{n_x\times 1}\\ \mathbf{v}\left(t+1\right)-\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right)\end{bmatrix}'\right)$$

$$=E\left(\begin{bmatrix}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\\ \mathbf{0}_{n_x\times 1}\\ \mathbf{v}\left(t+1\right)-\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right)\end{bmatrix}\begin{bmatrix}\sigma\boldsymbol{\epsilon}_{t+1}'\boldsymbol{\eta}' & \mathbf{0}_{1\times n_x} & \mathbf{v}'\left(t+1\right)-vec\left(\mathbf{I}_{n_e}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'\end{bmatrix}\right)$$

$$=E\begin{pmatrix}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\sigma\boldsymbol{\epsilon}_{t+1}'\boldsymbol{\eta}' & \mathbf{0}_{n_x\times n_x} & \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\mathbf{v}'\left(t+1\right)-vec\left(\mathbf{I}_{n_e}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'\right)\\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2}\\ \left(\mathbf{v}\left(t+1\right)-\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right)\right)\sigma\boldsymbol{\epsilon}_{t+1}'\boldsymbol{\eta}' & \mathbf{0}_{n_x\times n_x} & Var\left[\boldsymbol{\xi}_{t+1}\right]_{33}\end{pmatrix}$$

where

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{33}\equiv\left(\mathbf{v}\left(t+1\right)-\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right)\right)\left(\mathbf{v}'\left(t+1\right)-vec\left(\mathbf{I}_{n_e}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'\right)$$

Recall that

$$\mathbf{v}\left(t+1\right)=\left(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)$$

3.3.1 For $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{13}$

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{13}\equiv E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\mathbf{v}'\left(t+1\right)-vec\left(\mathbf{I}_{n_e}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'\right)]$$

$$=E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}(\left(\left(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right)'$$
$$-vec\left(\mathbf{I}_{n_e}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)')]$$

$$=E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}(\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\right)'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'$$
$$-vec\left(\mathbf{I}_{n_e}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'))]$$

$$=E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\right)'+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)'+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'$$

$$-\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} vec\left(\mathbf{I}_{n_e}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)']$$

$$= E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)']$$

because $E\left[\boldsymbol{\epsilon}_{t+1}\right]=\mathbf{0}$

$$= E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}'_{t+1}\otimes\boldsymbol{\epsilon}'_{t+1}\right)\left(\sigma\boldsymbol{\eta}'\otimes\boldsymbol{\sigma\eta}'\right)]$$

because $\boldsymbol{\epsilon}_{t+1}$ is independent of $\mathbf{x}_t^f$ and $E\left[\mathbf{x}_t^f\right]=0$. Hence, for shocks with a symmetry distribution $Var\left[\boldsymbol{\xi}_{t+1}\right]_{13}=\mathbf{0}$.

For the implementation, consider:
$E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}'_{t+1}\otimes\boldsymbol{\epsilon}'_{t+1}\right)\right]$

$$= E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{\epsilon_{t+1}\left(\phi_1,1\right)\right\}_{\phi_1=1}^{n_e}\left(\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{\epsilon_{t+1}\left(\phi_3,1\right)\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Hence the quasi MATLAB codes are :

```
E_eps_eps2 = zeros(ne, (ne)^2)
for phi1 = 1 : ne
    index2 = 0
    for phi2 = 1 : ne
        for phi3 = 1 : ne
            index2 = index2 + 1
            if (phi1 = phi2 = phi3)
                E_eps_eps2(phi1, index2) = m^3 (eps_{t+1} (phi1))
            end
        end
    end
end
```

3.3.2 For $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{33}$

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{33}\equiv E[\left(\mathbf{v}\left(t+1\right)-\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right)\right)\left(\mathbf{v}'\left(t+1\right)-vec\left(\mathbf{I}_{n_e}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'\right)]$$

$$= E[\left(\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)-\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right)\right)$$
$$\left(\left(\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right)'-vec\left(\mathbf{I}_{n_e}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'\right)]$$

$$= E[\left(\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)-vec\left(\mathbf{I}_{n_e}\right)\right)\right)$$
$$\left(\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)-vec\left(\mathbf{I}_{n_e}\right)\right)\right)']$$

$$= E[\left(\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)-vec\left(\mathbf{I}_{n_e}\right)\right)\right)$$
$$\left(\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'+\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'-vec\left(\mathbf{I}_{n_e}\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'\right)]$$

$$= E[\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\times$$

$$
\begin{aligned}
&\left(\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})' + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' + \left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' - vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'\right)\\
&+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) \times\\
&\left(\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})' + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' + \left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' - vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'\right)\\
&+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) - vec\left(\mathbf{I}_{n_e}\right)\right) \times\\
&\left(\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})' + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' + \left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' - vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'\right)]
\end{aligned}
$$

$$
\begin{aligned}
&= E[(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'\\
&+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'\\
&+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' - vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'\\
&+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'\\
&+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'\\
&+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' - vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'\\
&+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) - vec\left(\mathbf{I}_{n_e}\right)\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'\\
&+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) - vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'\\
&+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) - vec\left(\mathbf{I}_{n_e}\right)\right)\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' - vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})']
\end{aligned}
$$

$$
\begin{aligned}
&= E[(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\left(\mathbf{x}_t^f\right)' \otimes \boldsymbol{\epsilon}_{t+1}'\right)(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'\\
&+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}' \otimes \left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'\\
&+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})\left(\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' - \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'\\
&+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\left(\mathbf{x}_t^f\right)' \otimes \boldsymbol{\epsilon}_{t+1}'\right)(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'\\
&+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}' \otimes \left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'\\
&+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})\left(\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' - \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'\\
&+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) - vec\left(\mathbf{I}_{n_e}\right)\right)\left(\left(\mathbf{x}_t^f\right)' \otimes \boldsymbol{\epsilon}_{t+1}'\right)(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'\\
&+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) - vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1}' \otimes \left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'\\
&+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) - vec\left(\mathbf{I}_{n_e}\right)\right)\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' - vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})']
\end{aligned}
$$

$$
\begin{aligned}
&= E[(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)' \otimes \boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\right)(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'\\
&+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}' \otimes \boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'\\
&+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})\left(\left(\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}' \otimes \boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\right) - \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'\\
&+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)' \otimes \mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\right)(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'
\end{aligned}
$$

$$
\begin{aligned}
&+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'\\
&+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\right)-\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'\\
&+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\left((\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\left(\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}_{t+1}'\right)-vec\left(\mathbf{I}_{n_e}\right)\left(\left(\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}_{t+1}'\right)\right)(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'\\
&+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\left((\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\right)'\right)-vec\left(\mathbf{I}_{n_e}\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\right)'\right)\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'\\
&+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\left((\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\left(\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}'\right)-vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})']
\end{aligned}
$$

$$
\begin{aligned}
&=(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\otimes\mathbf{I}_{n_e}\right)(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'\\
&+(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})E\left(\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'\\
&+(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})E\left(\left(\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\right)-\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'\\
&+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})E\left(\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\otimes\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\right)(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'\\
&+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'\\
&+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})E\left(\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\right)-\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'\\
&+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})E\left((\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\left(\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}_{t+1}'\right)-vec\left(\mathbf{I}_{n_e}\right)\left(\left(\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}_{t+1}'\right)\right)(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'\\
&+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})E\left((\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\right)'\right)-vec\left(\mathbf{I}_{n_e}\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\right)'\right)\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'\\
&+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})E\left((\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\left(\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}'\right)-vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'
\end{aligned}
$$

note that $\boldsymbol{\epsilon}_{t+1}$ is independent of $\mathbf{x}_t^f$ and $E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]$ is know and $E\left[\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\right]=\mathbf{I}_{n_e}$

$$
\begin{aligned}
&=(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\otimes\mathbf{I}_{n_e}\right)(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'\\
&+(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})E\left(\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'\\
&+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})E\left(\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\otimes\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\right)(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'\\
&+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'\\
&+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})E\left((\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\left(\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}'\right)-vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'
\end{aligned}
$$

terms with three or one $\boldsymbol{\epsilon}_{t+1}$ are zero because $\mathbf{x}_t^f$ is independent of $\boldsymbol{\epsilon}_{t+1}$ and $E\left[\mathbf{x}_t^f\right]=0$

$$
\begin{aligned}
&=(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\otimes\mathbf{I}_{n_e}\right)(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'\\
&+(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})E\left(\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'\\
&+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})E\left(\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\otimes\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\right)(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'\\
&+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'\\
&+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\times
\end{aligned}
$$

$$E\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}'\right)-\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)vec\left(\mathbf{I}_{n_e}\right)'-vec\left(\mathbf{I}_{n_e}\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}'\right)+vec\left(\mathbf{I}_{n_e}\right)vec\left(\mathbf{I}_{n_e}\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'$$

$$\begin{aligned}
&=\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\otimes\mathbf{I}_{n_e}\right)\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'\\
&+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)E\left(\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'\\
&+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)E\left(\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\otimes\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\right)\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'\\
&+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'\\
&+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\times\\
&\left(E\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}'\right)-vec\left(\mathbf{I}_{n_e}\right)vec\left(\mathbf{I}_{n_e}\right)'-vec\left(\mathbf{I}_{n_e}\right)vec\left(\mathbf{I}_{n_e}\right)'+vec\left(\mathbf{I}_{n_e}\right)vec\left(\mathbf{I}_{n_e}\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'
\end{aligned}$$

$$\begin{aligned}
&=\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\otimes\mathbf{I}_{n_e}\right)\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'\\
&+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)E\left(\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'\\
&+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)E\left(\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\otimes\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\right)\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'\\
&+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'\\
&+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(E\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}'\right)-vec\left(\mathbf{I}_{n_e}\right)vec\left(\mathbf{I}_{n_e}\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'
\end{aligned}$$

Next consider (with dimensions $n_xn_e\times n_en_x$)

$$E\left[\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\right]$$

$$\begin{aligned}
&=E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\right)'\right)\right]\\
&=E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\left(\mathbf{x}_t^f\right)\right)'\right]\\
&=E\left[\left\{x_t^f\left(\gamma_1,1\right)\boldsymbol{\epsilon}_{t+1}\right\}_{\gamma_1=1}^{n_x}\left(\left\{\epsilon_{t+1}\left(\phi_2,1\right)\mathbf{x}_t^f\right\}_{\phi_2=1}^{n_e}\right)'\right]\\
&=E\left[\left\{x_t^f\left(\gamma_1,1\right)\left\{\epsilon_{t+1}\left(\phi_1,1\right)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\left(\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{x_t^f\left(\gamma_2,1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)'\right]
\end{aligned}$$

Thus the quasi Matlab codes are

```
E_xfeps_epsxf = zeros(nx ne, nx ne)
index1 = 0
for gama1 = 1 : nx
    for phi1 = 1 : ne
        index1 = index1 + 1
        index2 = 0
        for phi2 = 1 : ne
            for gama2 = 1 : nx
                index2 = index2 + 1
                if phi1 = phi2
                    E_xfeps_epsxf (index1, index2) = E_xf_xf(gama1, gama2)
```

end
end
end
end
end
where $E\_xf\_xf = reshape(E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right], nx, nx)$

Note also that

$$E\left[\left(\mathbf{x}_t^f \boldsymbol{\epsilon}'_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\right)'\right]$$

$$= E\left[\left(\mathbf{x}_t^f \boldsymbol{\epsilon}'_{t+1}\right)' \otimes \left(\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\right)'\right]$$

$$= E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)' \otimes \mathbf{x}_t^f \boldsymbol{\epsilon}'_{t+1}\right]$$

$\Updownarrow$

$(E\_xfeps\_epsxf)' = E\_epsxf\_xfeps$

Finally consider the matrix (with dimension $n_e^2 \times n_e^2$)

$$E\left[\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\right)\right]$$

$$= E[\begin{bmatrix} \epsilon_{t+1}(1,1) \\ \epsilon_{t+1}(2,1) \\ ... \\ \epsilon_{t+1}(n_e,1) \end{bmatrix} \begin{bmatrix} \epsilon'_{t+1}(1,1) & \epsilon'_{t+1}(1,2) & ... & \epsilon'_{t+1}(1,n_e) \end{bmatrix}$$
$$\otimes \begin{bmatrix} \epsilon_{t+1}(1,1) \\ \epsilon_{t+1}(2,1) \\ ... \\ \epsilon_{t+1}(n_e,1) \end{bmatrix} \begin{bmatrix} \epsilon'_{t+1}(1,1) & \epsilon'_{t+1}(1,2) & ... & \epsilon'_{t+1}(1,n_e) \end{bmatrix}]$$

$$= E[\begin{bmatrix} \epsilon_{t+1}(1,1)\epsilon'_{t+1}(1,1) & \epsilon_{t+1}(1,1)\epsilon'_{t+1}(1,2) & ... & \epsilon_{t+1}(1,1)\epsilon'_{t+1}(1,n_e) \\ \epsilon_{t+1}(2,1)\epsilon'_{t+1}(1,1) & \epsilon_{t+1}(2,1)\epsilon'_{t+1}(1,2) & ... & \epsilon_{t+1}(2,1)\epsilon'_{t+1}(1,n_e) \\ ... & ... & ... & ... \\ \epsilon_{t+1}(n_e,1)\epsilon_{t+1}(1,1) & \epsilon_{t+1}(n_e,1)\epsilon_{t+1}(1,2) & ... & \epsilon_{t+1}(n_e,1)\epsilon'_{t+1}(1,n_e) \end{bmatrix}$$
$$\otimes \begin{bmatrix} \epsilon_{t+1}(1,1)\epsilon'_{t+1}(1,1) & \epsilon_{t+1}(1,1)\epsilon'_{t+1}(1,2) & ... & \epsilon_{t+1}(1,1)\epsilon'_{t+1}(1,n_e) \\ \epsilon_{t+1}(2,1)\epsilon'_{t+1}(1,1) & \epsilon_{t+1}(2,1)\epsilon'_{t+1}(1,2) & ... & \epsilon_{t+1}(2,1)\epsilon'_{t+1}(1,n_e) \\ ... & ... & ... & ... \\ \epsilon_{t+1}(n_e,1)\epsilon_{t+1}(1,1) & \epsilon_{t+1}(n_e,1)\epsilon_{t+1}(1,2) & ... & \epsilon_{t+1}(n_e,1)\epsilon'_{t+1}(1,n_e) \end{bmatrix}]$$

$$= E[\begin{bmatrix} \epsilon_{t+1}(1,1)\epsilon'_{t+1}(1,1)\mathbf{A}_{\epsilon\epsilon} & \epsilon_{t+1}(1,1)\epsilon'_{t+1}(1,2)\mathbf{A}_{\epsilon\epsilon} & ... & \epsilon_{t+1}(1,1)\epsilon'_{t+1}(1,n_e)\mathbf{A}_{\epsilon\epsilon} \\ \epsilon_{t+1}(2,1)\epsilon_{t+1}(1,1)\mathbf{A}_{\epsilon\epsilon} & \epsilon_{t+1}(2,1)\epsilon'_{t+1}(1,2)\mathbf{A}_{\epsilon\epsilon} & ... & \epsilon_{t+1}(2,1)\epsilon'_{t+1}(1,n_e)\mathbf{A}_{\epsilon\epsilon} \\ ... & ... & ... & ... \\ \epsilon_{t+1}(n_e,1)\epsilon_{t+1}(1,1)\mathbf{A}_{\epsilon\epsilon} & \epsilon_{t+1}(n_e,1)\epsilon'_{t+1}(1,2)\mathbf{A}_{\epsilon\epsilon} & ... & \epsilon_{t+1}(n_e,1)\epsilon'_{t+1}(1,n_e)\mathbf{A}_{\epsilon\epsilon} \end{bmatrix}$$

where $\mathbf{A}_{\epsilon\epsilon} \equiv \begin{bmatrix} \epsilon_{t+1}(1,1)\epsilon'_{t+1}(1,1) & \epsilon_{t+1}(1,1)\epsilon'_{t+1}(1,2) & ... & \epsilon_{t+1}(1,1)\epsilon'_{t+1}(1,n_e) \\ \epsilon_{t+1}(2,1)\epsilon'_{t+1}(1,1) & \epsilon_{t+1}(2,1)\epsilon'_{t+1}(1,2) & ... & \epsilon_{t+1}(2,1)\epsilon'_{t+1}(1,n_e) \\ ... & ... & ... & ... \\ \epsilon_{t+1}(n_e,1)\epsilon_{t+1}(1,1) & \epsilon_{t+1}(n_e,1)\epsilon_{t+1}(1,2) & ... & \epsilon_{t+1}(n_e,1)\epsilon'_{t+1}(1,n_e) \end{bmatrix}$

$$= E[\begin{bmatrix} \epsilon_{t+1}(1,1)\,\epsilon'_{t+1}(1,1)\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\}_{\phi_1=1,\phi_2=1}^{n_e,n_e} & \epsilon_{t+1}(1,1)\,\epsilon'_{t+1}(1,2)\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\}_{\phi_1=1,\phi_2=1}^{n_e,n_e} & \ldots & \epsilon_{t+1}(1,1)\,\epsilon'_{t+1}(1,n_e)\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\}_{\phi_1=1,\phi_2=1}^{n_e,n_e} \\ \epsilon_{t+1}(2,1)\,\epsilon'_{t+1}(1,1)\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\}_{\phi_1=1,\phi_2=1}^{n_e,n_e} & \epsilon_{t+1}(2,1)\,\epsilon'_{t+1}(1,2)\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\}_{\phi_1=1,\phi_2=1}^{n_e,n_e} & \ldots & \epsilon_{t+1}(2,1)\,\epsilon'_{t+1}(1,n_e)\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\}_{\phi_1=1,\phi_2=1}^{n_e,n_e} \\ \ldots & \ldots & \ldots & \ldots \\ \epsilon_{t+1}(n_e,1)\,\epsilon'_{t+1}(1,1)\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\}_{\phi_1=1,\phi_2=1}^{n_e,n_e} & \epsilon_{t+1}(n_e,1)\,\epsilon'_{t+1}(1,2)\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\}_{\phi_1=1,\phi_2=1}^{n_e,n_e} & \ldots & \epsilon_{t+1}(n_e,1)\,\epsilon'_{t+1}(1,n_e)\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\}_{\phi_1=1,\phi_2=1}^{n_e,n_e} \end{bmatrix}$$

$$= E[\begin{bmatrix} \epsilon_{t+1}(1,1)\left\{\epsilon'_{t+1}(1,\phi_3)\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\}_{\phi_1=1,\phi_2=1}^{n_e,n_e}\right\}_{\phi_3=1}^{n_e} \\ \epsilon_{t+1}(2,1)\left\{\epsilon'_{t+1}(1,\phi_3)\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\}_{\phi_1=1,\phi_2=1}^{n_e,n_e}\right\}_{\phi_3=1}^{n_e} \\ \ldots \\ \epsilon_{t+1}(n_e,1)\left\{\epsilon'_{t+1}(1,\phi_3)\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\}_{\phi_1=1,\phi_2=1}^{n_e,n_e}\right\}_{\phi_3=1}^{n_e} \end{bmatrix}$$

Hence the quasi MATLAB codes are

```
E_eps2_eps2 = zeros(n_e^2, n_e^2)
index1 = 0
for phi4 = 1 : n_e
    for phi1 = 1 : n_e
        index1 = index1 + 1
        index2 = 0
        for phi3 = 1 : n_e
            for phi2 = 1 : n_e
                index2 = index2 + 1
                % second moments
                if (phi1 == phi2 && phi3 == phi4 && phi1~ = phi4)
                    E_eps2_eps2(index1, index2) = 1
                elseif (phi1 == phi3 && phi2 == phi4 && phi1~ = phi2)
                    E_eps2_eps2(index1, index2) = 1
                elseif (phi1 == phi4 && phi2 == phi3 && phi1~ = phi2)
                    E_eps2_eps2(index1, index2) = 1
                % fourth moments
                elseif(phi1 == phi2 && phi1 == phi3 && phi1 == phi4)
                    E_eps2_eps2(index1, index2) = m^4 (eps_{t+1} (phi1))
                end
            end
        end
    end
end
```

3.4 Method 3: Simple formulas for first and second moments

This section computes first and second moments at second order using a more direct approach. The advantage of this method is that we do not in a second-order approximation re-compute some of the moments already know from a first-order approximation. A direct implication is that we in Method 3 only need to invert smaller matrices than in Method 1 and 2.

3.4.1 First moments

Note first that

$$E[\mathbf{x}_t] = E\left[\mathbf{x}_t^f\right] + E[\mathbf{x}_t^s]$$

For the first order effects, we have due to stationary of the linear model

$E\left[\mathbf{x}_t^f\right] = \mathbf{h_x} E\left[\mathbf{x}_{t-1}^f\right] + \sigma \boldsymbol{\eta} E[\boldsymbol{\epsilon}_t]$

$\Updownarrow$

$\mathbf{I} E\left[\mathbf{x}_t^f\right] - \mathbf{h_x} E\left[\mathbf{x}_{t-1}^f\right] = E[\boldsymbol{\epsilon}_t]$

$\Updownarrow$

$(\mathbf{I} - \mathbf{h_x}) E\left[\mathbf{x}_t^f\right] = E[\boldsymbol{\epsilon}_t]$

$\Updownarrow$

$E\left[\mathbf{x}_t^f\right] = (\mathbf{I} - \mathbf{h_x})^{-1} E[\boldsymbol{\epsilon}_t]$

$\Updownarrow$

$E\left[\mathbf{x}_t^f\right] = \mathbf{0}$

since $E[\boldsymbol{\epsilon}_t] = \mathbf{0}$

For the second order effects we have

$E\left[x_{t+1}^s(j,1)\right] = \mathbf{h_x}(j,:) E[\mathbf{x}_t^s] + \frac{1}{2} E\left[\left(\mathbf{x}_t^f\right)' \mathbf{h_{xx}}(j,:,:) \left(\mathbf{x}_t^f\right)\right] + \frac{1}{2} h_{\sigma\sigma}(j,1)\sigma^2$

$\Updownarrow$

$(\mathbf{I} - \mathbf{h_x}) E[\mathbf{x}_t^s] = \tilde{\mathbf{H}}_{\mathbf{xx}} E\left[vec\left(\left[\left(\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right]\right)\right] + \frac{1}{2} \mathbf{h}_{\sigma\sigma}\sigma^2$

where $\tilde{\mathbf{H}}_{\mathbf{xx}} \equiv \frac{1}{2} reshape\left(\mathbf{h_{xx}}, n_x, n_x^2\right)$

To compute $E\left[\mathbf{x}_t^f \left(\mathbf{x}_t^f\right)'\right]$, consider

$Var\left(\mathbf{x}_t^f\right) = \mathbf{h_x} Var\left(\mathbf{x}_t^f\right) \mathbf{h}_\mathbf{x}' + \sigma^2 \boldsymbol{\eta}\boldsymbol{\eta}'$

$\Downarrow$

$vec\left(Var\left(\mathbf{x}_t^f\right)\right) = vec\left(\mathbf{h_x} Var\left(\mathbf{x}_t^f\right) \mathbf{h}_\mathbf{x}'\right) + vec\left(\sigma^2 \boldsymbol{\eta}\boldsymbol{\eta}'\right)$

$\Updownarrow$

$vec\left(Var\left(\mathbf{x}_t^f\right)\right) = (\mathbf{h_x} \otimes \mathbf{h_x}) vec\left(Var\left(\mathbf{x}_t^f\right)\right) + vec\left(\sigma^2 \boldsymbol{\eta}\boldsymbol{\eta}'\right)$

$\Updownarrow$

$\left(\mathbf{I}_{n_x^2} - (\mathbf{h_x} \otimes \mathbf{h_x})\right) vec\left(Var\left(\mathbf{x}_t^f\right)\right) = vec\left(\sigma^2 \boldsymbol{\eta}\boldsymbol{\eta}'\right)$

$\Updownarrow$

$vec\left(Var\left(\mathbf{x}_t^f\right)\right) = \left(\mathbf{I}_{n_x^2} - (\mathbf{h_x} \otimes \mathbf{h_x})\right)^{-1} vec\left(\sigma^2 \boldsymbol{\eta}\boldsymbol{\eta}'\right)$

Notice there that

$Var\left(\mathbf{x}_t^f\right) = E\left[\left(\mathbf{x}_t^f - E\left[\mathbf{x}_t^f\right]\right)\left(\mathbf{x}_t^f - E\left[\mathbf{x}_t^f\right]\right)'\right] = E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]$

since $E\left[\mathbf{x}_t^f\right] = \mathbf{0}$

$\Updownarrow$

$vec\left(E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\right) = vec\left(Var\left(\mathbf{x}_t^f\right)\right)$

Hence,

$E\left[\mathbf{x}_t^s\right] = \left(\mathbf{I} - \mathbf{h_x}\right)^{-1}\left(\tilde{\mathbf{H}}_{\mathbf{xx}}E\left[vec\left(\left[\left(\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right]\right)\right] + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)$

The mean value of the control variables is given by

$E\left[\mathbf{y}_t^s\right] = \mathbf{g_x}\left(E\left[\mathbf{x}_t^f\right] + E\left[\mathbf{x}_t^s\right]\right) + \tilde{\mathbf{G}}_{\mathbf{xx}}E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$

$\Updownarrow$

$E\left[\mathbf{y}_t^s\right] = \mathbf{g_x}E\left[\mathbf{x}_t^s\right] + \tilde{\mathbf{G}}_{\mathbf{xx}}E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$

3.4.2 Second moments

We need to compute $E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$, $E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$, $E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^s\right)'\right]$, $E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$ and $E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^s\right)'\right]$ as this will allow us to find $Var\left(\mathbf{z}_t\right)$. This is because

$$Var\left(\mathbf{z}_t\right) = E\left[\left(\mathbf{z}_t - E\left[\mathbf{z}_t\right]\right)\left(\mathbf{z}_t - E\left[\mathbf{z}_t\right]\right)'\right]$$

$$= E\left[\left(\mathbf{z}_t - E\left[\mathbf{z}_t\right]\right)\left(\mathbf{z}_t' - E\left[\mathbf{z}_t'\right]\right)\right]$$

$$= E\left[\mathbf{z}_t\mathbf{z}_t' - \mathbf{z}_tE\left[\mathbf{z}_t'\right] - E\left[\mathbf{z}_t\right]\mathbf{z}_t' + E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\right]$$

$$= E\left[\mathbf{z}_t\mathbf{z}_t'\right] - E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]$$

and

$$E\left[\mathbf{z}_t\mathbf{z}_t'\right] = E\left[\begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{bmatrix}\begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^s\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \end{bmatrix}\right]$$

$$= E\begin{bmatrix} \mathbf{x}_t^f\left(\mathbf{x}_t^f\right)' & \mathbf{x}_t^f\left(\mathbf{x}_t^s\right)' & \mathbf{x}_t^f\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \\ \mathbf{x}_t^s\mathbf{x}_t^f & \mathbf{x}_t^s\left(\mathbf{x}_t^s\right)' & \mathbf{x}_t^s\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \\ \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\mathbf{x}_t^f & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^s\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \end{bmatrix}$$

Finding $E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$

From above:

$\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f = \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \mathbf{v}\left(t+1\right)$

where

$\mathbf{v}\left(t+1\right) = \left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) + \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)$

and $E\left[\mathbf{v}\left(t+1\right)\right] = \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right)$. Note that $\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)$ and $\mathbf{v}\left(t+1\right)'$ are uncorrelated

So

$$E\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)'$$

$$=E\{(\mathbf{h_x}\otimes\mathbf{h_x})\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\mathbf{v}\left(t+1\right)\}\{(\mathbf{h_x}\otimes\mathbf{h_x})\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\mathbf{v}\left(t+1\right)\}'$$

$$=E\{(\mathbf{h_x}\otimes\mathbf{h_x})\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\mathbf{v}\left(t+1\right)\}\{\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'(\mathbf{h_x}\otimes\mathbf{h_x})'+\mathbf{v}\left(t+1\right)'\}$$

$$=E\left[(\mathbf{h_x}\otimes\mathbf{h_x})\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'(\mathbf{h_x}\otimes\mathbf{h_x})'+\mathbf{v}\left(t+1\right)'\right)\right]$$
$$+E\left[\mathbf{v}\left(t+1\right)\left(\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'(\mathbf{h_x}\otimes\mathbf{h_x})'+\mathbf{v}\left(t+1\right)'\right)\right]$$

$$=E\left[(\mathbf{h_x}\otimes\mathbf{h_x})\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'(\mathbf{h_x}\otimes\mathbf{h_x})'+(\mathbf{h_x}\otimes\mathbf{h_x})\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\mathbf{v}\left(t+1\right)'\right]$$
$$+E\left[\mathbf{v}\left(t+1\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'(\mathbf{h_x}\otimes\mathbf{h_x})'+\mathbf{v}\left(t+1\right)\mathbf{v}\left(t+1\right)'\right]$$

$$=(\mathbf{h_x}\otimes\mathbf{h_x})\,E\left[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\right](\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})\,E\left[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\right]E\left[\mathbf{v}\left(t+1\right)'\right]+E\left[\mathbf{v}\left(t+1\right)\right]E\left[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\right](\mathbf{h_x}\otimes\mathbf{h_x})'+E\left[\mathbf{v}\left(t+1\right)\mathbf{v}\left(t+1\right)'\right]$$

Letting

$\mathbf{c}\equiv(\mathbf{h_x}\otimes\mathbf{h_x})\,E\left[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\right]E\left[\mathbf{v}\left(t+1\right)'\right]+E\left[\mathbf{v}\left(t+1\right)\right]E\left[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\right](\mathbf{h_x}\otimes\mathbf{h_x})'+E\left[\mathbf{v}\left(t+1\right)\mathbf{v}\left(t+1\right)'\right]$

we therefore have (due to stationarity)

$E[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)']=(\mathbf{h_x}\otimes\mathbf{h_x})\,E[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'](\mathbf{h_x}\otimes\mathbf{h_x})'+\mathbf{c}$

$\Updownarrow$

$$vec\left(E[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)']\right)=vec\left((\mathbf{h_x}\otimes\mathbf{h_x})\,E[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'](\mathbf{h_x}\otimes\mathbf{h_x})'\right)+vec\left(\mathbf{c}\right)$$

$\Updownarrow$

$$vec\left(E[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)']\right)=((\mathbf{h_x}\otimes\mathbf{h_x})\otimes(\mathbf{h_x}\otimes\mathbf{h_x}))\,vec\left(E[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)']\right)+vec\left(\mathbf{c}\right)$$

because $vec\left(\mathbf{ABC}\right)=\left(\mathbf{C}'\otimes\mathbf{A}\right)vec\left(\mathbf{B}\right)$

$\Updownarrow$

$$vec\left(E[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)']\right)\left(\mathbf{I}_{n_x^4}-(\mathbf{h_x}\otimes\mathbf{h_x})\otimes(\mathbf{h_x}\otimes\mathbf{h_x})\right)=vec\left(\mathbf{c}\right)$$

$\Updownarrow$

$$vec\left(E[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)']\right)=vec\left(\mathbf{c}\right)\left(\mathbf{I}_{n_x^4}-(\mathbf{h_x}\otimes\mathbf{h_x})\otimes(\mathbf{h_x}\otimes\mathbf{h_x})\right)^{-1}$$

Finding $E\left[\mathbf{x}_{t}^{s}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\right]$

$$E\left[\mathbf{x}_{t+1}^{s}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)'\right]=E\left[\left(\mathbf{h_x}\mathbf{x}_{t}^{s}+\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\left(\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\mathbf{v}\left(t+1\right)\right)'\right]$$

$$=E\left[\left(\mathbf{h_x}\mathbf{x}_{t}^{s}+\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\left(\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)'+\mathbf{v}\left(t+1\right)'\right)\right]$$

$$=E\left[\mathbf{h_x}\mathbf{x}_{t}^{s}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)'+\mathbf{h_x}\mathbf{x}_{t}^{s}\mathbf{v}\left(t+1\right)'\right]$$
$$+E\left[\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)'+\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\mathbf{v}\left(t+1\right)'\right]$$
$$+E\left[\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)'+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\mathbf{v}\left(t+1\right)'\right]$$

$$=\mathbf{h_x}E\left[\mathbf{x}_{t}^{s}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\right]\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)'+\mathbf{h_x}E\left[\mathbf{x}_{t}^{s}\right]E\left[\mathbf{v}\left(t+1\right)'\right]$$
$$+\tilde{\mathbf{H}}_{\mathbf{xx}}E\left[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\right]\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)'+\tilde{\mathbf{H}}_{\mathbf{xx}}E\left[\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right]E\left[\mathbf{v}\left(t+1\right)'\right]$$
$$+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}E\left[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\right]\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)'+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}E\left[\mathbf{v}\left(t+1\right)'\right]$$

Letting

$$\mathbf{c}\equiv\mathbf{h_x}E\left[\mathbf{x}_{t}^{s}\right]E\left[\mathbf{v}\left(t+1\right)'\right]+\tilde{\mathbf{H}}_{\mathbf{xx}}E\left[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\right]\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)'+\tilde{\mathbf{H}}_{\mathbf{xx}}E\left[\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right]E\left[\mathbf{v}\left(t+1\right)'\right]$$
$$+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}E\left[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\right]\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)'+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}E\left[\mathbf{v}\left(t+1\right)'\right]$$

we therefore have (due to stationarity)

$$E\left[\mathbf{x}_{t+1}^{s}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)'\right]=\mathbf{h_x}E\left[\mathbf{x}_{t}^{s}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\right]\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)'+\mathbf{c}$$

$\Updownarrow$

$$vec\left(E\left[\mathbf{x}_{t+1}^{s}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)'\right]\right)=vec\left(\mathbf{h_x}E\left[\mathbf{x}_{t}^{s}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\right]\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)'\right)+vec\left(\mathbf{c}\right)$$

$\Updownarrow$

$$vec\left(E\left[\mathbf{x}_{t+1}^{s}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)'\right]\right)=\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)vec\left(E\left[\mathbf{x}_{t}^{s}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\right]\right)+vec\left(\mathbf{c}\right)$$

$\Updownarrow$

$$vec\left(E\left[\mathbf{x}_{t}^{s}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\right]\right)\left(\mathbf{I}_{n_x^3}-\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)\right)=vec\left(\mathbf{c}\right)$$

$\Updownarrow$

$$vec\left(E\left[\mathbf{x}_{t}^{s}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\right]\right)=vec\left(\mathbf{c}\right)\left(\mathbf{I}_{n_x^3}-\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)\right)^{-1}$$

Finding $E\left[\mathbf{x}_{t}^{s}\left(\mathbf{x}_{t}^{s}\right)'\right]$

$$E\left[\mathbf{x}_{t+1}^{s}\left(\mathbf{x}_{t+1}^{s}\right)'\right]=E\left[\left(\mathbf{h_x}\mathbf{x}_{t}^{s}+\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\left(\mathbf{h_x}\mathbf{x}_{t}^{s}+\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)'\right]$$

$$=E\left[\left(\mathbf{h_x}\mathbf{x}_{t}^{s}+\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\left(\left(\mathbf{x}_{t}^{s}\right)'\mathbf{h}_{\mathbf{x}}'+\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\tilde{\mathbf{H}}_{\mathbf{xx}}'+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^{2}\right)\right]$$

$$= E\left[\mathbf{h_x}\mathbf{x}_t^s\left(\left(\mathbf{x}_t^s\right)'\mathbf{h}_\mathbf{x}' + \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\tilde{\mathbf{H}}_{\mathbf{xx}}' + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2\right)\right]$$
$$+E\left[\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\left(\mathbf{x}_t^s\right)'\mathbf{h}_\mathbf{x}' + \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\tilde{\mathbf{H}}_{\mathbf{xx}}' + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2\right)\right]$$
$$+E\left[\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\left(\left(\mathbf{x}_t^s\right)'\mathbf{h}_\mathbf{x}' + \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\tilde{\mathbf{H}}_{\mathbf{xx}}' + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2\right)\right]$$

$$= E\left[\mathbf{h_x}\mathbf{x}_t^s\left(\mathbf{x}_t^s\right)'\mathbf{h}_\mathbf{x}' + \mathbf{h_x}\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\tilde{\mathbf{H}}_{\mathbf{xx}}' + \mathbf{h_x}\mathbf{x}_t^s\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2\right]$$
$$+E\left[\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^s\right)'\mathbf{h}_\mathbf{x}' + \tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\tilde{\mathbf{H}}_{\mathbf{xx}}' + \tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2\right]$$
$$+E\left[\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\left(\mathbf{x}_t^s\right)'\mathbf{h}_\mathbf{x}' + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\tilde{\mathbf{H}}_{\mathbf{xx}}' + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2\right]$$

$$= \mathbf{h_x}E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^s\right)'\right]\mathbf{h}_\mathbf{x}' + \mathbf{h_x}E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\tilde{\mathbf{H}}_{\mathbf{xx}}' + \mathbf{h_x}E\left[\mathbf{x}_t^s\right]\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2$$
$$+\tilde{\mathbf{H}}_{\mathbf{xx}}E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^s\right)'\right]\mathbf{h}_\mathbf{x}' + \tilde{\mathbf{H}}_{\mathbf{xx}}E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\tilde{\mathbf{H}}_{\mathbf{xx}}' + \tilde{\mathbf{H}}_{\mathbf{xx}}E\left[\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right]\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2$$
$$+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2E\left[\left(\mathbf{x}_t^s\right)'\right]\mathbf{h}_\mathbf{x}' + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\tilde{\mathbf{H}}_{\mathbf{xx}}' + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2$$

Letting

$$\mathbf{c}\equiv\mathbf{h_x}E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\tilde{\mathbf{H}}_{\mathbf{xx}}' + \mathbf{h_x}E\left[\mathbf{x}_t^s\right]\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2$$
$$+\tilde{\mathbf{H}}_{\mathbf{xx}}E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^s\right)'\right]\mathbf{h}_\mathbf{x}' + \tilde{\mathbf{H}}_{\mathbf{xx}}E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\tilde{\mathbf{H}}_{\mathbf{xx}}' + \tilde{\mathbf{H}}_{\mathbf{xx}}E\left[\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right]\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2$$
$$+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2E\left[\left(\mathbf{x}_t^s\right)'\right]\mathbf{h}_\mathbf{x}' + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\tilde{\mathbf{H}}_{\mathbf{xx}}' + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2$$

we therefore have (due to stationarity)

$$E\left[\mathbf{x}_{t+1}^s\left(\mathbf{x}_{t+1}^s\right)'\right] = \mathbf{h_x}E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^s\right)'\right]\mathbf{h}_\mathbf{x}' + \mathbf{c}$$
$$\Updownarrow$$
$$E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^s\right)'\right] = vec\left(c\right)\left(\mathbf{I}_{n_x^2} - \left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\right)^{-1}$$

Finding $E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]$

$$E\left[\mathbf{x}_{t+1}^f\left(\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right)'\right]$$

$$= E\left[\left(\mathbf{h_x}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right) + \mathbf{v}\left(t+1\right)\right)'\right]$$

$$= E\left[\left(\mathbf{h_x}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)' + \mathbf{v}\left(t+1\right)'\right)\right]$$

$$= E\left[\mathbf{h_x}\mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)' + \mathbf{h_x}\mathbf{x}_t^f\mathbf{v}\left(t+1\right)'\right]$$
$$+E\left[\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)' + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\mathbf{v}\left(t+1\right)'\right)\right]$$

Recall that $\mathbf{v}\left(t+1\right) = \left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right) + \left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)$ so we get

$$= \mathbf{h_x} E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right](\mathbf{h_x} \otimes \mathbf{h_x})' + \mathbf{0} + E\left[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'\right]$$

$$= \mathbf{h_x} E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right](\mathbf{h_x} \otimes \mathbf{h_x})' + \mathbf{0} + \sigma\boldsymbol{\eta} E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right](\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'$$

$\Updownarrow$

$$vec\left(E\left[\mathbf{x}_{t+1}^f\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right)'\right]\right) = (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \mathbf{h_x})\, vec\left(E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\right)$$
$$+vec\left(\sigma\boldsymbol{\eta} E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right](\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'\right)$$

because $vec(\mathbf{ABC}) = (\mathbf{C}' \otimes \mathbf{A})\, vec(\mathbf{B})$

$\Updownarrow$

$$\left(E\left[\mathbf{x}_{t+1}^f\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right)'\right]\right) = \left(\mathbf{I}_{n_x^3} - (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \mathbf{h_x})\right)^{-1} vec\left(\sigma\boldsymbol{\eta} E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right](\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'\right)$$

Finding $E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^s\right)'\right]$

$$E\left[\mathbf{x}_{t+1}^f\left(\mathbf{x}_{t+1}^s\right)'\right] = E\left[\left(\mathbf{h_x}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{h_x}\mathbf{x}_t^s + \tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right]$$

$$= E\left[\mathbf{h_x}\mathbf{x}_t^f\left(\left(\mathbf{x}_t^s\right)'\mathbf{h}_\mathbf{x}' + \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\tilde{\mathbf{H}}_{\mathbf{xx}}' + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2\right)\right]$$

$$= \mathbf{h_x} E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^s\right)'\right]\mathbf{h}_\mathbf{x}' + \mathbf{h_x} E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\tilde{\mathbf{H}}_{\mathbf{xx}}'$$

$\Updownarrow$

$$vec\left(E\left[\mathbf{x}_{t+1}^f\left(\mathbf{x}_{t+1}^s\right)'\right]\right) = (\mathbf{h_x} \otimes \mathbf{h_x})\, vec\left(E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^s\right)'\right]\right) + vec\left(\mathbf{h_x} E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\tilde{\mathbf{H}}_{\mathbf{xx}}'\right)$$

$\Updownarrow$

$$vec\left(E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^s\right)'\right]\right) = \left(\mathbf{I}_{n_x^2} - (\mathbf{h_x} \otimes \mathbf{h_x})\right)^{-1} vec\left(\mathbf{h_x} E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\tilde{\mathbf{H}}_{\mathbf{xx}}'\right)$$

3.5 The auto-correlations

This section derives the auto-correlations for the states and the control variables.

3.5.1 The innovations

We start by showing that $\boldsymbol{\xi}_{t+1}$ and $\boldsymbol{\xi}_{t+1+s}$ are uncorrelated for $s = 1, 2, ...$ To see this note that
$E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1+s}'\right] =$

$$E\left[\left[\begin{array}{c}\boldsymbol{\epsilon}_{t+1}\\ \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} - vec(\mathbf{I}_{n_e})\\ \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\\ \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\end{array}\right] \begin{array}{cccc}\boldsymbol{\epsilon}_{t+1+s}' & \left(\boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s} - vec(\mathbf{I}_{n_e})\right)' & \left(\boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f\right)' & \left(\mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s}\right)'\end{array}\right]$$

$$= E\left[\begin{array}{cc} \boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1+s}' & \boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}-vec\left(\mathbf{I}_{n_e}\right)\right)' \\ \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)\boldsymbol{\epsilon}_{t+1+s}' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}-vec\left(\mathbf{I}_{n_e}\right)\right)' \\ \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\boldsymbol{\epsilon}_{t+1+s}' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}-vec\left(\mathbf{I}_{n_e}\right)\right)' \\ \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}_{t+1+s}' & \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}-vec\left(\mathbf{I}_{n_e}\right)\right)' \end{array}\right.$$

$$\left.\begin{array}{cc} \boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\right)' & \boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \\ \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \\ \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \\ \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\right)' & \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \end{array}\right]$$

$$= \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

3.5.2 The auto-covariances

Recall that we have

$$\mathbf{z}_t = \left[\begin{array}{c} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f\otimes\mathbf{x}_t^f \end{array}\right]$$

$$\mathbf{z}_{t+1} = \mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}$$

$$\mathbf{y}_t = \mathbf{D}\mathbf{z}_t + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$$

To find the one period auto-correlation, i.e. $Cov\left(\mathbf{z}_{t+1},\mathbf{z}_t\right)$, we have

$$Cov\left(\mathbf{z}_{t+1},\mathbf{z}_t\right) = Cov\left(\mathbf{c}+\mathbf{A}\mathbf{z}_t+\mathbf{B}\boldsymbol{\xi}_{t+1},\mathbf{z}_t\right) = \mathbf{A}Cov\left(\mathbf{z}_t,\mathbf{z}_t\right) = \mathbf{A}Var\left(\mathbf{z}_t\right)$$

because $Cov\left(\mathbf{z}_t,\boldsymbol{\xi}_{t+1}\right)=0$ as shown above. And for two periods

$Cov\left(\mathbf{z}_{t+2},\mathbf{z}_t\right) = Cov\left(\mathbf{c}+\mathbf{A}\mathbf{z}_{t+1}+\mathbf{B}\boldsymbol{\xi}_{t+2},\mathbf{z}_t\right)$
$= Cov\left(\mathbf{A}\left(\mathbf{c}+\mathbf{A}\mathbf{z}_t+\mathbf{B}\boldsymbol{\xi}_{t+1}\right)+\mathbf{B}\boldsymbol{\xi}_{t+2},\mathbf{z}_t\right)$
$= Cov\left(\mathbf{A}^2\mathbf{z}_t+\mathbf{B}\boldsymbol{\xi}_{t+2},\mathbf{z}_t\right)$
$= Cov\left(\mathbf{A}^2\mathbf{z}_t,\mathbf{z}_t\right)$
$= \mathbf{A}^2 Cov\left(\mathbf{z}_t,\mathbf{z}_t\right)$
$= \mathbf{A}^2 Var\left(\mathbf{z}_t\right)$

Here, we use the fact that $Cov\left(\boldsymbol{\xi}_{t+2},\mathbf{z}_t\right)=0$. This follows from the same arguements as above, that is consider

$$E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+2}'\right] = E\left[\left[\begin{array}{c} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f\otimes\mathbf{x}_t^f \end{array}\right]\left[\begin{array}{cccc} \boldsymbol{\epsilon}_{t+2}' & \left(\boldsymbol{\epsilon}_{t+2}\otimes\boldsymbol{\epsilon}_{t+2}-vec\left(\mathbf{I}_{ne}\right)\right)' & \left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^f\right)' & \left(\mathbf{x}_{t+1}^f\otimes\boldsymbol{\epsilon}_{t+2}\right)' \end{array}\right]\right]$$

$$= E\left[\begin{array}{cccc} \mathbf{x}_t^f\boldsymbol{\epsilon}_{t+2}' & \mathbf{x}_t^f\left(\boldsymbol{\epsilon}_{t+2}\otimes\boldsymbol{\epsilon}_{t+2}-vec\left(\mathbf{I}_{ne}\right)\right)' & \mathbf{x}_t^f\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^f\right)' & \mathbf{x}_t^f\left(\mathbf{x}_{t+1}^f\otimes\boldsymbol{\epsilon}_{t+2}\right)' \\ \mathbf{x}_t^s\boldsymbol{\epsilon}_{t+2}' & \mathbf{x}_t^s\left(\boldsymbol{\epsilon}_{t+2}\otimes\boldsymbol{\epsilon}_{t+2}-vec\left(\mathbf{I}_{ne}\right)\right)' & \mathbf{x}_t^s\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^f\right)' & \mathbf{x}_t^s\left(\mathbf{x}_{t+1}^f\otimes\boldsymbol{\epsilon}_{t+2}\right)' \\ \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\boldsymbol{\epsilon}_{t+1}' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\boldsymbol{\epsilon}_{t+2}-vec\left(\mathbf{I}_{ne}\right)\right)' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^f\right)' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_{t+1}^f\otimes\boldsymbol{\epsilon}_{t+2}\right)' \end{array}\right]$$

$= 0$

Hence, in the general case
$Cov(\mathbf{z}_{t+l}, \mathbf{z}_t) = \mathbf{A}^l Var(\mathbf{z}_t)$

For the control variables:
$Cov(\mathbf{y}_{t+l}, \mathbf{y}_t) = Cov\left(\mathbf{D}\mathbf{z}_{t+l} + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2, \mathbf{D}\mathbf{z}_t + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2\right)$

$= Cov(\mathbf{D}\mathbf{z}_{t+l}, \mathbf{D}\mathbf{z}_t)$

$= \mathbf{D}Cov(\mathbf{z}_{t+l}, \mathbf{z}_t)\mathbf{D}'$

$= \mathbf{D}\mathbf{A}^l Var(\mathbf{z}_t)\mathbf{D}'$

4 Stastical properties: Third order approximation

4.1 Covariance-stationary

Proposition 1:
The pruned third order approximation for $\mathbf{x}_t^f, \mathbf{x}_t^s, \mathbf{x}_t^{rd}$, and $\mathbf{y}_t^{rd}$ is covariance-stationary if

1. the DSGE model has a unique stable equilibrium, i.e. all eigenvalue of $\mathbf{h}_\mathbf{x}$ have modulus less than 1
2. $\boldsymbol{\epsilon}_{t+1}$ has finite sixth moment

Proof
Note first that

$$x_{t+1}^{rd}(j,1) = \mathbf{h}_\mathbf{x}(j,:)\mathbf{x}_t^{rd} + \frac{2}{2}\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)(\mathbf{x}_t^s)$$
$$+\frac{1}{6}\left(\mathbf{x}_t^f\right)'\begin{bmatrix} \left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,1,:,:)\left(\mathbf{x}_t^f\right) \\ \dots \\ \left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,n_x,:,:)\left(\mathbf{x}_t^f\right) \end{bmatrix} + \frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}(j,:)\sigma^2\mathbf{x}_t^f + \frac{1}{6}h_{\sigma\sigma\sigma}(j,1)\sigma^3$$

$\Updownarrow$

$$\mathbf{x}_{t+1}^{rd} = \mathbf{h}_\mathbf{x}\mathbf{x}_t^{rd} + 2\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \tilde{\mathbf{H}}_{\mathbf{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f + \frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

where $\tilde{\mathbf{H}}_{\mathbf{xx}} \equiv \frac{1}{2}reshape\left(\mathbf{h}_{\mathbf{xx}}, n_x, n_x^2\right)$ and $\tilde{\mathbf{H}}_{\mathbf{xxx}} \equiv \frac{1}{6}reshape\left(\mathbf{h}_{\mathbf{xxx}}, n_x, n_x^3\right)$. So we need to find the law of motion of $\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)$ and $\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)$.

Hence,

$$\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s\right) = \left(\mathbf{h}_\mathbf{x}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes \left(\mathbf{h}_\mathbf{x}\mathbf{x}_t^s + \tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)$$

$$= \mathbf{h}_\mathbf{x}\mathbf{x}_t^f \otimes \mathbf{h}_\mathbf{x}\mathbf{x}_t^s + \mathbf{h}_\mathbf{x}\mathbf{x}_t^f \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \mathbf{h}_\mathbf{x}\mathbf{x}_t^f \otimes \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$
$$+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_\mathbf{x}\mathbf{x}_t^s + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$

using $(\mathbf{A}+\mathbf{B})\otimes(\mathbf{C}+\mathbf{D}) = \mathbf{A}\otimes\mathbf{C} + \mathbf{A}\otimes\mathbf{D} + \mathbf{B}\otimes\mathbf{C} + \mathbf{B}\otimes\mathbf{D}$

$$= (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \left(\mathbf{h_x} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\right) \left(\mathbf{x}_t^f \otimes \sigma^2\right)$$
$$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) (\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s) + \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \sigma^2\right)$$

using $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}$

$$= (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \left(\mathbf{h_x} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \mathbf{x}_t^f$$
$$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) (\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s) + \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \boldsymbol{\epsilon}_{t+1}$$

Recall from above

$$\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f = (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)$$
$$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})$$

So

$$\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f = \left(\mathbf{h_x}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes \left((\mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\right.$$
$$\left. + (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})\right)$$

$$= \mathbf{h_x}\mathbf{x}_t^f \otimes (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \mathbf{h_x}\mathbf{x}_t^f \otimes (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)$$
$$+ \mathbf{h_x}\mathbf{x}_t^f \otimes (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + \mathbf{h_x}\mathbf{x}_t^f \otimes (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})$$
$$+ \left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)$$
$$+ \left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})$$

$$= (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)$$
$$+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)$$
$$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)$$
$$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})$$

$$= (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)$$
$$+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)$$
$$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)$$
$$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta}) \left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) - E\left[(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})\right]\right)$$
$$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta}) E\left[(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})\right]$$

Thus we can construct the following extended system

$$\begin{bmatrix} \mathbf{x}_{t+1}^f \\ \mathbf{x}_{t+1}^s \\ \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \\ \mathbf{x}_{t+1}^{rd} \\ \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s \\ \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n_x \times 1} \\ \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\, vec\left(\mathbf{I}_{n_e}\right) \\ \frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3 \\ \mathbf{0}_{n_x^2 \times 1} \\ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta}) E\left[(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})\right] \end{bmatrix}$$

$$
+\left[\begin{array}{cccccc}
\mathbf{h_x} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x^3}\\
\mathbf{0}_{n_x\times n_x} & \mathbf{h_x} & \tilde{\mathbf{H}}_{\mathbf{xx}} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x^3}\\
\mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x} & \mathbf{h_x}\otimes\mathbf{h_x} & \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x^2} & \mathbf{0}_{n_x^2\times n_x^3}\\
\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2 & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{h_x} & 2\tilde{\mathbf{H}}_{\mathbf{xx}} & \tilde{\mathbf{H}}_{\mathbf{xxx}}\\
\left(\mathbf{h_x}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) & \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x^2} & \mathbf{0}_{n_x^2\times n_x} & (\mathbf{h_x}\otimes\mathbf{h_x}) & \left(\mathbf{h_x}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\\
\mathbf{0}_{n_x^3\times n_x} & \mathbf{0}_{n_x^3\times n_x} & \mathbf{0}_{n_x^3\times n_x^2} & \mathbf{0}_{n_x^3\times n_x} & \mathbf{0}_{n_x^3\times n_x^2} & (\mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x})
\end{array}\right]
\left[\begin{array}{c}
\mathbf{x}_t^f\\ \mathbf{x}_t^s\\ \mathbf{x}_t^f\otimes\mathbf{x}_t^f\\ \mathbf{x}_t^{rd}\\ \mathbf{x}_t^f\otimes\mathbf{x}_t^s\\ \mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f
\end{array}\right]
$$

$$
+\left[\begin{array}{cccccccccccc}
\sigma\boldsymbol{\eta} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & (\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}) & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x} & \mathbf{h_x}\otimes\sigma\boldsymbol{\eta} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x} & \sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x} & \mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta} & \mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x} & \mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x} & \sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}
\end{array}\right]
$$

$$
\times\left[\begin{array}{c}
\boldsymbol{\epsilon}_{t+1}\\
\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\\
\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\\
\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\\
\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\\
\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\\
\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\\
\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\\
\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\\
\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\\
\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\\
(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})-E\left[(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\right]
\end{array}\right]
$$

$\Updownarrow$

$$
\mathbf{z}_{t+1}=\mathbf{c}+\mathbf{A}\mathbf{z}_t+\mathbf{B}\boldsymbol{\xi}_{t+1} \tag{38}
$$

The absolute value of the eigenvalues in $\mathbf{h_x}$ are all strictly less then one by assumption. Accordingly, all eigenvalues of $\mathbf{A}$ are also strictly less than one. To see this note first that

$p(\lambda)=|\mathbf{A}-\lambda\mathbf{I}|$

$$
=\left|\left[\begin{array}{cccccc}
\mathbf{h_x}-\lambda\mathbf{I} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x^3}\\
\mathbf{0}_{n_x\times n_x} & \mathbf{h_x}-\lambda\mathbf{I} & \tilde{\mathbf{H}}_{\mathbf{xx}} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x^3}\\
\mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x} & \mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I} & \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x^2} & \mathbf{0}_{n_x^2\times n_x^3}\\
\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2 & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{h_x}-\lambda\mathbf{I} & 2\tilde{\mathbf{H}}_{\mathbf{xx}} & \tilde{\mathbf{H}}_{\mathbf{xxx}}\\
\mathbf{h_x}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 & \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x^2} & \mathbf{0}_{n_x^2\times n_x} & \mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I} & \mathbf{h_x}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\\
\mathbf{0}_{n_x^3\times n_x} & \mathbf{0}_{n_x^3\times n_x} & \mathbf{0}_{n_x^3\times n_x^2} & \mathbf{0}_{n_x^3\times n_x} & \mathbf{0}_{n_x^3\times n_x^2} & \mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I}
\end{array}\right]\right|
$$

$$
=\left|\left[\begin{array}{cc}
\mathbf{B}_{11} & \mathbf{B}_{12}\\
\mathbf{B}_{21} & \mathbf{B}_{22}
\end{array}\right]\right|
$$

where we let

$$\mathbf{B}_{11} \equiv \begin{bmatrix} \mathbf{h_x}-\lambda\mathbf{I} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{h_x}-\lambda\mathbf{I} & \tilde{\mathbf{H}}_{\mathbf{xx}} \\ \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x} & \mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I} \end{bmatrix}$$

$$\mathbf{B}_{12} \equiv \begin{bmatrix} \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x^3} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x^3} \\ \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x^2} & \mathbf{0}_{n_x^2\times n_x^3} \end{bmatrix}$$

$$\mathbf{B}_{21} \equiv \begin{bmatrix} \frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2 & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} \\ \left(\mathbf{h_x}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) & \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x^2} \\ \mathbf{0}_{n_x^3\times n_x} & \mathbf{0}_{n_x^3\times n_x} & \mathbf{0}_{n_x^3\times n_x^2} \end{bmatrix}$$

$$\mathbf{B}_{22} \equiv \begin{bmatrix} \mathbf{h_x}-\lambda\mathbf{I} & 2\tilde{\mathbf{H}}_{\mathbf{xx}} & \tilde{\mathbf{H}}_{\mathbf{xxx}} \\ \mathbf{0}_{n_x^2\times n_x} & (\mathbf{h_x}\otimes\mathbf{h_x})-\lambda\mathbf{I} & \left(\mathbf{h_x}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right) \\ \mathbf{0}_{n_x^3\times n_x} & \mathbf{0}_{n_x^3\times n_x^2} & (\mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x})-\lambda\mathbf{I} \end{bmatrix}$$

$= |\mathbf{B}_{11}|\,|\mathbf{B}_{22}|$

using $\begin{vmatrix} \mathbf{U} & \mathbf{C} \\ \mathbf{0} & \mathbf{Y} \end{vmatrix} = |\mathbf{U}|\,|\mathbf{Y}|$ where $\mathbf{U}$ is $m\times m$ and $\mathbf{Y}$ is $n\times n$

$= |\mathbf{h_x}-\lambda\mathbf{I}|\,|\mathbf{h_x}-\lambda\mathbf{I}|\,|\mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I}|\,|\mathbf{B}_{22}|$

using the results from the second order approximation

$= |\mathbf{h_x}-\lambda\mathbf{I}|\,|\mathbf{h_x}-\lambda\mathbf{I}|\,|\mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I}|\,|\mathbf{h_x}-\lambda\mathbf{I}|\,|(\mathbf{h_x}\otimes\mathbf{h_x})-\lambda\mathbf{I}|\,|(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x})-\lambda\mathbf{I}|$

using the rule on block determinants repeatedly on $\mathbf{B}_{22}$

Hence, the eigenvalue λ solves the problem

$p(\lambda)=0$

$\Updownarrow$

$|\mathbf{h_x}-\lambda\mathbf{I}|\,|\mathbf{h_x}-\lambda\mathbf{I}|\,|\mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I}|\,|\mathbf{h_x}-\lambda\mathbf{I}|\,|(\mathbf{h_x}\otimes\mathbf{h_x})-\lambda\mathbf{I}|\,|(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x})-\lambda\mathbf{I}|=0$

$\Updownarrow$

$|\mathbf{h_x}-\lambda\mathbf{I}|=0$ or $|\mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I}|=0$ or $|(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x})-\lambda\mathbf{I}|=0$

The absolute value of all eigenvalues to the first problem are strictly less than one. That is $|\lambda_i|<1$ $i=1,2,...,n_x$. This is also the case for the second problem because the eigenvalues to $\mathbf{h_x}\otimes\mathbf{h_x}$ are $\lambda_i\lambda_j$ for $i=1,2,...,n_x$ and $j=1,2,...,n_x$. The same arguement ensures that this is also the case for the third problem.

Thus, the system in (38) is covariance stationary if $\boldsymbol{\xi}_{t+1}$ has finite first and second moment. It follows directly that $E\left[\boldsymbol{\xi}_{t+1}\right]=\mathbf{0}$ and $\boldsymbol{\xi}_{t+1}$ has finite second moments if $\boldsymbol{\epsilon}_{t+1}$ has a sixth moment. The latter holds by assumption.

For the control variables we have

$$y_t^{rd}(i,1)=\mathbf{g_x}(i,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\frac{1}{2}\left(\mathbf{x}_t^f\right)'\mathbf{g_{xx}}(i,:,:)\left(\mathbf{x}_t^f+2\mathbf{x}_t^s\right)$$
$$+\frac{1}{6}\left(\mathbf{x}_t^f\right)'\begin{bmatrix} \left(\mathbf{x}_t^f\right)'\mathbf{g_{xxx}}(i,1,:,:)\left(\mathbf{x}_t^f\right) \\ ... \\ \left(\mathbf{x}_t^f\right)'\mathbf{g_{xxx}}(i,n_x,:,:)\left(\mathbf{x}_t^f\right) \end{bmatrix}+\frac{1}{2}g_{\sigma\sigma}(i,1)\sigma^2+\frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}(i,:)\sigma^2\mathbf{x}_t^f+\frac{1}{6}g_{\sigma\sigma\sigma}(i,1)\sigma^3$$

$\Updownarrow$

$$\mathbf{y}_t^{rd}=\mathbf{g_x}\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\tilde{\mathbf{G}}_{\mathbf{xx}}\left(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+2\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\right)+\tilde{\mathbf{G}}_{\mathbf{xxx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)$$
$$+\frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2+\frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f+\frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$

where $\tilde{\mathbf{G}}_{\mathbf{xx}}\equiv\frac{1}{2}reshape\left(\mathbf{g_{xx}},n_y,n_x^2\right)$ and $\tilde{\mathbf{G}}_{\mathbf{xxx}}\equiv\frac{1}{6}reshape\left(\mathbf{g_{xxx}},n_y,n_x^3\right)$

$\Updownarrow$

$$\mathbf{y}_t^{rd} = \begin{bmatrix} \mathbf{g_x} + \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2 & \mathbf{g_x} & \tilde{\mathbf{G}}_{\mathbf{xx}} & \mathbf{g_x} & 2\tilde{\mathbf{G}}_{\mathbf{xx}} & \tilde{\mathbf{G}}_{\mathbf{xxx}} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \\ \mathbf{x}_t^{rd} \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{bmatrix} + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$

$$= \mathbf{D}\mathbf{z}_t + \tfrac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \tfrac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$

That is $\mathbf{y}_t^{rd}$ is linear function of $\mathbf{z}_t$ and $\mathbf{y}_t^{rd}$ is therefore also covariance-stationary.

Q.E.D.

4.2 Method 1: Formulas for the first and second moments

This section computes first and second moments using the representation of the second-order system stated above. This method is fairly direct but has the computational disadvantage of requiring a lot of memory because we work directly with the big $\mathbf{B}$ matrix.

The system

$$\mathbf{z}_{t+1} = \mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}$$

$$\mathbf{y}_t^{rd} = \mathbf{D}\mathbf{z}_t + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$

The mean values are

$$E[\mathbf{z}_t] = \left(\mathbf{I}_{3n_x+2n_x^2+n_x^3} - \mathbf{A}\right)^{-1}\mathbf{c}.$$

$$E\left[\mathbf{y}_t^{rd}\right] = \mathbf{D}E[\mathbf{z}_t] + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$

For the variances we first have as above that

$$E\left[\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right] = E\left[\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}\right)\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}\right)'\right]$$

$$= E\left[\mathbf{c}\mathbf{c}' + \mathbf{c}\mathbf{z}_t'\mathbf{A}' + \mathbf{c}\boldsymbol{\xi}_{t+1}'\mathbf{B}'\right]$$
$$+E\left[\mathbf{A}\mathbf{z}_t\mathbf{c}' + \mathbf{A}\mathbf{z}_t\mathbf{z}_t'\mathbf{A}' + \mathbf{A}\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\mathbf{B}'\right]$$
$$+E\left[\mathbf{B}\boldsymbol{\xi}_{t+1}\mathbf{c}' + \mathbf{B}\boldsymbol{\xi}_{t+1}\mathbf{z}_t'\mathbf{A}' + \mathbf{B}\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\mathbf{B}'\right]$$

$$= \mathbf{c}\mathbf{c}' + \mathbf{c}E[\mathbf{z}_t']\mathbf{A}'$$
$$+\mathbf{A}E[\mathbf{z}_t]\mathbf{c}' + \mathbf{A}E[\mathbf{z}_t\mathbf{z}_t']\mathbf{A}' + \mathbf{A}E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$
$$+\mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$

and

$$E[\mathbf{z}_t]E[\mathbf{z}_t]' = (\mathbf{c} + \mathbf{A}E[\mathbf{z}_t])(\mathbf{c} + \mathbf{A}E[\mathbf{z}_t])'$$

$$= (\mathbf{c} + \mathbf{A}E[\mathbf{z}_t])\mathbf{c}' + \mathbf{c}E[\mathbf{z}_t']\mathbf{A}' + \mathbf{A}E[\mathbf{z}_t]E[\mathbf{z}_t']\mathbf{A}'$$

Hence,

$$E\left[\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right] - E[\mathbf{z}_t]E[\mathbf{z}_t]' = \mathbf{c}\mathbf{c}' + \mathbf{c}E[\mathbf{z}_t']\mathbf{A}'$$
$$+\mathbf{A}E[\mathbf{z}_t]\mathbf{c}' + \mathbf{A}E[\mathbf{z}_t\mathbf{z}_t']\mathbf{A}' + \mathbf{A}E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$
$$+\mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$
$$-(\mathbf{c} + \mathbf{A}E[\mathbf{z}_t])\mathbf{c}' - \mathbf{c}E[\mathbf{z}_t']\mathbf{A}' - \mathbf{A}E[\mathbf{z}_t]E[\mathbf{z}_t']\mathbf{A}$$

$$= \mathbf{A}\left(E\left[\mathbf{z}_t\mathbf{z}_t'\right] - E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\right)\mathbf{A}'$$
$$+\mathbf{A}E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\mathbf{z}_t'\right]\mathbf{A}'$$
$$+\mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$
$\Updownarrow$

$$Var\left[\mathbf{z}_{t+1}\right] = \mathbf{A}Var[\mathbf{z_t}]\,\mathbf{A}'$$
$$+\mathbf{A}\left(E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\right] - E\left[\mathbf{z}_t\right]E\left[\boldsymbol{\xi}_{t+1}'\right]\right)\mathbf{B}' + \mathbf{B}\left(E\left[\boldsymbol{\xi}_{t+1}\mathbf{z}_t'\right] - E\left[\boldsymbol{\xi}_{t+1}\right]E\left[\mathbf{z}_t'\right]\right)\mathbf{A}'$$
$$+\mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$
Notice that $E\left[\mathbf{z}_t\right]E\left[\boldsymbol{\xi}_{t+1}'\right] = 0$ because $E\left[\boldsymbol{\xi}_{t+1}'\right] = 0$
$\Updownarrow$

$$Var\left[\mathbf{z}_{t+1}\right] = \mathbf{A}Var[\mathbf{z_t}]\,\mathbf{A}' + \mathbf{B}Var\left[\boldsymbol{\xi}_{t+1}\right]\mathbf{B}' + \mathbf{A}Cov\left[\mathbf{z}_t, \boldsymbol{\xi}_{t+1}\right]\mathbf{B}' + \mathbf{B}Cov\left[\boldsymbol{\xi}_{t+1}, \mathbf{z}_t\right]\mathbf{A}'$$

Moreover,
$$Var\left[\mathbf{y}_t^{rd}\right] = \mathbf{D}Var\left[\mathbf{z}_t\right]\mathbf{D}'$$

Contrary to a second-order approximation, we have that $Cov\left[\mathbf{z}_t, \boldsymbol{\xi}_{t+1}\right] \neq 0$. This is seen as follows

$$E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\right] = E\left[\begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \\ \mathbf{x}_t^{rd} \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{bmatrix}\right.$$

$$\times\left[\begin{array}{l} \boldsymbol{\epsilon}_{t+1}' \quad \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)' \quad \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' \quad \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)' \quad \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)' \quad \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)' \\ \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)' \quad \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' \quad \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)' \quad \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)' \\ \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' \quad \left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right) - E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right]\right)' \end{array}\right]$$

$$= \begin{bmatrix} 0_{n_x\times n_e} & 0_{n_x\times n_e^2} & 0_{n_x\times n_e n_x} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_e n_x^2} & 0_{n_x\times n_x^2 n_e} & 0_{n_x\times n_x^2 n_e} & r_{1,9} & r_{1,10} & r_{1,11} & 0_{n_x\times n_e^3} \\ 0_{n_x\times n_e} & 0_{n_x\times n_e^2} & 0_{n_x\times n_e n_x} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_e n_x^2} & 0_{n_x\times n_x^2 n_e} & 0_{n_x\times n_x^2 n_e} & r_{2,9} & r_{2,10} & r_{2,11} & 0_{n_x\times n_e^3} \\ 0_{n_x^2\times n_e} & 0_{n_x^2\times n_e^2} & 0_{n_x^2\times n_e n_x} & 0_{n_x^2\times n_x n_e} & 0_{n_x^2\times n_x n_e} & 0_{n_x^2\times n_e n_x^2} & 0_{n_x^2\times n_x^2 n_e} & 0_{n_x^2\times n_x^2 n_e} & r_{3,9} & r_{3,10} & r_{3,11} & 0_{n_x^2\times n_e^3} \\ 0_{n_x\times n_e} & 0_{n_x\times n_e^2} & 0_{n_x\times n_e n_x} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_e n_x^2} & 0_{n_x\times n_x^2 n_e} & 0_{n_x\times n_x^2 n_e} & r_{4,9} & r_{4,10} & r_{4,11} & 0_{n_x\times n_e^3} \\ 0_{n_x^2\times n_e} & 0_{n_x^2\times n_e^2} & 0_{n_x^2\times n_e n_x} & 0_{n_x^2\times n_x n_e} & 0_{n_x^2\times n_x n_e} & 0_{n_x^2\times n_e n_x^2} & 0_{n_x^2\times n_x^2 n_e} & 0_{n_x^2\times n_x^2 n_e} & r_{5,9} & r_{5,10} & r_{5,1} & 0_{n_x^2\times n_e^3} \\ 0_{n_x^3\times n_e} & 0_{n_x^3\times n_e^2} & 0_{n_x^3\times n_e n_x} & 0_{n_x^3\times n_x n_e} & 0_{n_x^3\times n_x n_e} & 0_{n_x^3\times n_e n_x^2} & 0_{n_x^3\times n_x^2 n_e} & 0_{n_x^3\times n_x^2 n_e} & r_{6,9} & r_{6,10} & r_{6,11} & 0_{n_x^3\times n_e^3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \mathbf{R} & 0 \end{bmatrix}$$

We now compute the non-zero elements in this matrix

1) The value of $r_{1,9}$
$$r_{1,9} = E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^f\left(\gamma_1,1\right)\right\}_{\gamma_1=1}^{n_x}\left\{x_t^f\left(\gamma_2,1\right)\left\{\epsilon_{t+1}\left(\phi_1,1\right)\left\{\epsilon_{t+1}\left(\phi_2,1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right]$$
Thus, the quasi Matlab codes are
$E\_xf\_xfeps2 = zeros(nx, nx\times ne\times ne)$
$for\ gama1 = 1 : nx$

```
    index2 = 0
    for gama2 = 1 : nx
        for phi1 = 1 : ne
            for phi2 = 1 : ne
                index2 = index2 + 1
                if phi1 == phi2
                    E_xf_xfeps2(phi1, index2) = E_xf_xf(gama1, gama2)
                end
            end
        end
    end
end
```

2) The value of $r_{1,10}$

$$r_{1,10} = E\left[\mathbf{x}_t^f \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^f\left(\gamma_1, 1\right)\right\}_{\gamma_1=1}^{n_x} \left\{\epsilon_{t+1}\left(\phi_1, 1\right) \left\{x_t^f\left(\gamma_2, 1\right) \left\{\epsilon_{t+1}\left(\phi_2, 1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right]$$

3) The value of $r_{1,11}$

$$r_{1,11} = E\left[\mathbf{x}_t^f \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$$

$$= E\left[\left\{x_t^f\left(\gamma_1, 1\right)\right\}_{\gamma_1=1}^{n_x} \left\{\epsilon_{t+1}\left(\phi_1, 1\right) \left\{\epsilon_{t+1}\left(\phi_2, 1\right) \left\{x_t^f\left(\gamma_2, 1\right)\right\}_{\gamma_2}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right]$$

4) The value of $r_{2,9}$

$$r_{2,9} = E\left[\mathbf{x}_t^s \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^s\left(\gamma_1, 1\right)\right\}_{\gamma_1=1}^{n_x} \left\{x_t^f\left(\gamma_2, 1\right) \left\{\epsilon_{t+1}\left(\phi_1, 1\right) \left\{\epsilon_{t+1}\left(\phi_2, 1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right]$$

5) The value of $r_{1,10}$

$$r_{2,10} = E\left[\mathbf{x}_t^s \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^s\left(\gamma_1, 1\right)\right\}_{\gamma_1=1}^{n_x} \left\{\epsilon_{t+1}\left(\phi_1, 1\right) \left\{x_t^f\left(\gamma_2, 1\right) \left\{\epsilon_{t+1}\left(\phi_2, 1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right]$$

6) The value of $r_{2,11}$

$$r_{2,11} = E\left[\mathbf{x}_t^s \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$$

$$= E\left[\left\{x_t^s\left(\gamma_1, 1\right)\right\}_{\gamma_1=1}^{n_x} \left\{\epsilon_{t+1}\left(\phi_1, 1\right) \left\{\epsilon_{t+1}\left(\phi_2, 1\right) \left\{x_t^f\left(\gamma_2, 1\right)\right\}_{\gamma_2}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right]$$

7) The value of $r_{3,9}$

$$r_{3,9} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right]$$

8) The value of $r_{3,10}$

$$r_{3,10} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right]$$

9) The value of $r_{3,11}$

$$r_{3,11} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_3,1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right]$$

10) The value of $r_{4,9}$

$$r_{4,9} = E\left[\mathbf{x}_t^{rd}\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^{rd}(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right]$$

11) The value of $r_{4,10}$

$$r_{4,10} = E\left[\mathbf{x}_t^{rd}\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^{rd}(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right]$$

12) The value of $r_{4,11}$

$$r_{4,11} = E\left[\mathbf{x}_t^{rd}\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$$

$$= E\left[\left\{x_t^{rd}(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right]$$

13) The value of $r_{5,9}$

$$r_{5,9} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^s(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right]$$

14) The value of $r_{5,10}$

$$r_{5,10} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^s(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right]$$

15) The value of $r_{5,11}$

$$r_{5,11} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^s(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_3,1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right]$$

16) The value of $r_{6,9}$

$$r_{6,9} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\left\{x_t^f(\gamma_3,1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{x_t^f(\gamma_4,1)\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_4=1}^{n_x}\right]$$

17) The value of $r_{6,10}$

$$r_{6,10} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\left\{x_t^f(\gamma_3,1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_4,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_4=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right]$$

18) The value of $r_{6,11}$

$$r_{6,11} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^s(\gamma_2,1)\left\{x_t^f(\gamma_3,1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_4,1)\right\}_{\gamma_4=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right]$$

Notice that all the required moments needed to compute these 18 terms are available from the covariance matrix at second order. Hence we only need to compute $Var\left(\boldsymbol{\xi}_{t+1}\right)$. This is done below.

4.2.1 Efficient computing of $\mathbf{B}Cov\left[\boldsymbol{\xi}_{t+1},\mathbf{z}_t\right]$

The matrix $\mathbf{B}$ is very big and we therefore by hand try to simplify the summations $\mathbf{B}Cov\left[\boldsymbol{\xi}_{t+1},\mathbf{z}_t\right]$. Note that such a simplified expression is also useful when computing auto-correlations. We first note that

$$\mathbf{B}Cov\left[\boldsymbol{\xi}_{t+1},\mathbf{z}_t\right]=\mathbf{B}E\left[\boldsymbol{\xi}_{\mathbf{t+1}}\mathbf{z}_{\mathbf{t}}'\right]$$

$$=\mathbf{B}\begin{bmatrix}0\\ \mathbf{R}'\\ 0\end{bmatrix}$$

because $E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\right]=\begin{bmatrix}0 & \mathbf{R} & 0\end{bmatrix}$

$$=\begin{bmatrix}
\sigma\boldsymbol{\eta} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & (\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}) & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x} & \mathbf{h_x}\otimes\sigma\boldsymbol{\eta} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x} & \sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x} & \mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta} & \mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x} & \mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x} & \sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}
\end{bmatrix}$$

$$\times\begin{bmatrix}0\\ \mathbf{R}'\\ 0\end{bmatrix}$$

$$=\begin{bmatrix}0\\0\\0\\0\\0\\ \begin{bmatrix}\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x}\end{bmatrix}\mathbf{R}'\end{bmatrix}$$

$$=\begin{bmatrix}0_{n_x\times(3n_x+2n_x^2+n_x^3)}\\0_{n_x\times(3n_x+2n_x^2+n_x^3)}\\0_{n_x^2\times(3n_x+2n_x^2+n_x^3)}\\0_{n_x\times(3n_x+2n_x^2+n_x^3)}\\0_{n_x^2\times(3n_x+2n_x^2+n_x^3)}\\ \begin{bmatrix}\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x}\end{bmatrix}\mathbf{R}'\end{bmatrix}$$

We see that $\begin{bmatrix}\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x}\end{bmatrix}$ has dimensions $n_x^3\times 3\left(n_xn_e^2\right)$ and $\mathbf{R}$ has dimensions $\left(3n_x+2n_x^2+n_x^3\right)\times\left(3n_x^2n_e\right)$. Thus

$\begin{bmatrix}\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x}\end{bmatrix}\mathbf{R}'$ has dimensions $n_x^3\times\left(3n_x+2n_x^2+n_x^3\right)$

Hence,

$$Cov\left[\mathbf{z}_t,\boldsymbol{\xi}_{t+1}\right]\mathbf{B}'$$

$$=\left(\mathbf{B}Cov\left[\boldsymbol{\xi}_{t+1},\mathbf{z}_t\right]\right)'$$

$$= \left[\begin{array}{cccccc} 0_{nn\times n_x} & 0_{nn\times n_x} & 0_{nn\times n_x^2} & 0_{nn\times n_x} & 0_{nn\times n_x^2} & \mathbf{R}\left[\begin{array}{ccc} \mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x} \end{array} \right]' \end{array} \right]$$

where $nn = \left(3n_x + 2n_x^2 + n_x^3\right)$

4.2.2 Computing $Var\left[\boldsymbol{\xi}_{t+1}\right]$

We start by noticing that

$$E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right] = E\left[\left[\begin{array}{c} \boldsymbol{\epsilon}_{t+1} \\ \boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right) \\ \boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f \\ \mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1} \\ \boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s \\ \boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f \\ \mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1} \\ \mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f \\ \mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} \\ \boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1} \\ \boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f \\ \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right) - E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right] \end{array}\right]\right.$$

$$\times\left[\begin{array}{l} \boldsymbol{\epsilon}_{t+1}' \quad \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)' \quad \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' \quad \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)' \quad \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)' \quad \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)' \\ \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)' \quad \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' \quad \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)' \quad \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)' \\ \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' \quad \left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right) - E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right]\right)' \end{array} \right]$$

$$= \left[\begin{array}{cccccccccccc} p_{1,1} & p_{1,2} & 0 & 0 & p_{1,5} & p_{1,6} & p_{1,7} & p_{1,8} & 0 & 0 & 0 & p_{1,12} \\ & p_{2,2} & 0 & 0 & p_{2,5} & p_{2,6} & p_{2,7} & p_{2,8} & 0 & 0 & 0 & p_{2,12} \\ & & p_{3,3} & p_{3,4} & p_{3,5} & p_{3,6} & p_{3,7} & p_{3,8} & p_{3,9} & p_{3,10} & p_{3,11} & p_{3,12} \\ & & & p_{4,4} & p_{4,5} & p_{4,6} & p_{4,7} & p_{4,8} & p_{4,9} & p_{4,10} & p_{4,11} & p_{4,12} \\ & & & & p_{5,5} & p_{5,6} & p_{5,7} & p_{5,8} & p_{5,9} & p_{5,10} & p_{5,11} & p_{5,12} \\ & & & & & p_{6,6} & p_{6,7} & p_{6,8} & p_{6,9} & p_{6,10} & p_{6,11} & p_{6,12} \\ & & & & & & p_{7,7} & p_{7,8} & p_{7,9} & p_{7,10} & p_{7,11} & p_{7,12} \\ & & & & & & & p_{8,8} & p_{8,9} & p_{8,10} & p_{8,11} & p_{8,12} \\ & & & & & & & & p_{9,9} & p_{9,10} & p_{9,11} & p_{9,12} \\ & & & & & & & & & p_{10,10} & p_{10,11} & p_{10,12} \\ & & & & & & & & & & p_{11,11} & 0 \\ & & & & & & & & & & & p_{12,12} \end{array}\right]$$

Only stating the elements on and above the diagonal. We first notice that $E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right]$ can be computed as:

```
E_eps3 = zeros(ne × ne × ne, 1)
index = 0
for phi1 = 1 : ne
    for phi2 = 1 : ne
        for phi3 = 1 : ne
            index = index + 1
            if phi1 == phi2  &&  phi1 == phi3
                E_eps3(index, 1) = m^3 (ε_{t+1} (phi1))
            end
```

end

end

end

We next compute all the elements in this matrix. The method is illustrated below

1) for $p_{1,1}$

$E\left[\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\right] = \mathbf{I}$

2) for $p_{1,2}$

$$E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)'\right] = E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{\epsilon_{t+1}\left(\phi_1,1\right)\right\}_{\phi_1=1}^{n_e}\left(\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{\epsilon_{t+1}\left(\phi_3,1\right)\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Hence the quasi MATLAB codes are :

$E\_eps\_eps2 = zeros(ne,(ne)^2)$

$for\ phi1 = 1:ne$

$index2 = 0$

$for\ phi2 = 1:ne$

$for\ phi3 = 1:ne$

$index2 = index2 + 1$

$if\ (phi1 = phi2 = phi3)$

$E\_eps\_eps2(phi1,index2) = m^3\left(\epsilon_{t+1}\left(phi1\right)\right)$

end

end

end

end

3) for $p_{1,5}$

$$E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\right] = E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\right]$$

$$= E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes 1\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^s\right)'\right)\right]$$

$$= E\left[\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^s\right)'\right]$$

$$= \mathbf{I}\otimes E\left[\left(\mathbf{x}_t^s\right)'\right]$$

4) for $p_{1,6}$

$$E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right] = E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes 1\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\right]$$

$$= E\left[\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]$$

$$= \mathbf{I}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]$$

5) for $p_{1,7}$

$$E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right] = E\left[1\otimes\boldsymbol{\epsilon}_{t+1}\left(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}'_{t+1}\right)\right]$$

$$= E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\right]$$

$$= E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right] \otimes \mathbf{I}$$

6) for $p_{1,8}$

$$E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right] =$$

$$E\left[\left\{\epsilon_{t+1}(\phi_1, 1)\right\}_{\phi_1=1}^{n_e}\left(\left\{x_t^f(\gamma_1, 1)\left\{\epsilon_{t+1}(\phi_2, 1)\left\{x_t^f(\gamma_2, 1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)'\right]$$

Thus, the quasi Matlab codes are

```
E_eps_xfepsxf = zeros(ne, nx × ne × nx)
for phi1 = 1 : ne
    index2 = 0
    for phi2 = 1 : ne
        for gama1 = 1 : nx
            for phi2 = 1 : ne
                for gama2 = 1 : nx
                index2 = index2 + 1
                if phi1 == phi2
                    E_eps_xfepsxf(phi1, index2) = E_xf_xf(gama1, gama2)
                end
            end
        end
    end
end
```

....

4.3 Method 2: Formulas for the first and second moments

This section computes first and second moments using a slightly different representation of the third-order system than stated above. (Basically, this was the first representation we considered for computing these moments). The advantage of this method is that it compared to Method 1 is less memory intensive because some of the matrix multiplications are done by hand.

We first recall that

$$\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s\right) = (\mathbf{h_x} \otimes \mathbf{h_x})\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \left(\mathbf{h_x} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_t^f$$
$$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s\right) + \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}$$

We also know that

$$\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f = (\mathbf{h_x} \otimes \mathbf{h_x})\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)$$
$$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)$$

so

$$\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f = \left(\mathbf{h_x}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes \left((\mathbf{h_x} \otimes \mathbf{h_x})\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\right.$$
$$\left. + (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\right)$$

$$= \mathbf{h_x}\mathbf{x}_t^f \otimes (\mathbf{h_x} \otimes \mathbf{h_x})\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \mathbf{h_x}\mathbf{x}_t^f \otimes (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)$$
$$+ \mathbf{h_x}\mathbf{x}_t^f \otimes (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + \mathbf{h_x}\mathbf{x}_t^f \otimes (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)$$

$$+\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\otimes\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\otimes\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)$$

$$=\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\left(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)$$

$$=\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\mathbf{u}_{t+1}$$

where $\mathbf{u}_{t+1}\equiv\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)$

$$+\left(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)$$

Thus we can construct the following extended system

$$\begin{bmatrix}\mathbf{x}_{t+1}^f\\ \mathbf{x}_{t+1}^s\\ \mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\\ \mathbf{x}_{t+1}^{rd}\\ \mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^s\\ \mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\end{bmatrix}=\begin{bmatrix}\mathbf{h}_{\mathbf{x}} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x^3}\\ \mathbf{0}_{n_x\times n_x} & \mathbf{h}_{\mathbf{x}} & \tilde{\mathbf{H}}_{\mathbf{xx}} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x^3}\\ \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x} & \mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}} & \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x^2} & \mathbf{0}_{n_x^2\times n_x^3}\\ \frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2 & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{h}_{\mathbf{x}} & 2\tilde{\mathbf{H}}_{\mathbf{xx}} & \tilde{\mathbf{H}}_{\mathbf{xxx}}\\ \left(\mathbf{h}_{\mathbf{x}}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) & \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x^2} & \mathbf{0}_{n_x^2\times n_x} & \left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right) & \left(\mathbf{h}_{\mathbf{x}}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\\ \mathbf{0}_{n_x^3\times n_x} & \mathbf{0}_{n_x^3\times n_x} & \mathbf{0}_{n_x^3\times n_x^2} & \mathbf{0}_{n_x^3\times n_x} & \mathbf{0}_{n_x^3\times n_x^2} & \left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\end{bmatrix}$$
$$\times\begin{bmatrix}\mathbf{x}_t^f\\ \mathbf{x}_t^s\\ \mathbf{x}_t^f\otimes\mathbf{x}_t^f\\ \mathbf{x}_t^{rd}\\ \mathbf{x}_t^f\otimes\mathbf{x}_t^s\\ \mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\end{bmatrix}$$
$$+\begin{bmatrix}\mathbf{0}_{n_x\times 1}\\ \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\\ \left(\sigma\boldsymbol{\eta}\otimes\sigma\boldsymbol{\eta}\right)vec\left(\mathbf{I}_{n_e}\right)\\ +\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\\ \mathbf{0}_{n_x^2\times 1}\\ \mathbf{0}_{n_x^3\times 1}+E\left[\mathbf{u}_{t+1}\right]\end{bmatrix}+\begin{bmatrix}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\\ \mathbf{0}_{n_x\times 1}\\ \mathbf{v}\left(t+1\right)-\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right)\\ \mathbf{0}_{n_x\times 1}\\ \left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\\ \mathbf{u}_{t+1}-E\left[\mathbf{u}_{t+1}\right]\end{bmatrix}$$

$\Updownarrow$

$$\mathbf{z}_{t+1}=\mathbf{c}+\mathbf{A}\mathbf{z}_t+\tilde{\boldsymbol{\xi}}_{t+1}$$

Hence, $\tilde{\boldsymbol{\xi}}_{t+1}\equiv\mathbf{B}\boldsymbol{\xi}_{t+1}$. The expression for the controls are as before, i.e.

$$\mathbf{y}_t=\mathbf{D}\mathbf{z}_t+\frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2+\frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$

The mean values are

$$E\left[\mathbf{z}_t\right]=\left(\mathbf{I}_{3n_x+2n_x^2+n_x^3}-\mathbf{A}\right)^{-1}\mathbf{c}.$$

$$E\left[\mathbf{y}_t\right] = \mathbf{D}E\left[\mathbf{z}_t\right] + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$

We showed above that

$$Var\left[\mathbf{z}_{t+1}\right] = \mathbf{A}Var[\mathbf{z_t}]\,\mathbf{A}' + \mathbf{B}Var\left[\boldsymbol{\xi}_{t+1}\right]\mathbf{B}' + \mathbf{A}Cov\left[\mathbf{z}_t, \boldsymbol{\xi}_{t+1}\right]\mathbf{B}' + \mathbf{B}Cov\left[\boldsymbol{\xi}_{t+1}, \mathbf{z}_t\right]\mathbf{A}'$$

which is equivalent to

$$Var\left[\mathbf{z}_{t+1}\right] = \mathbf{A}Var[\mathbf{z_t}]\,\mathbf{A}' + Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right] + \mathbf{A}Cov\left[\mathbf{z}_t, \boldsymbol{\xi}_{t+1}\right]\mathbf{B}' + \mathbf{B}Cov\left[\boldsymbol{\xi}_{t+1}, \mathbf{z}_t\right]\mathbf{A}'$$

We have already known how to compute the $\mathbf{A}Cov\left[\mathbf{z}_t, \boldsymbol{\xi}_{t+1}\right]\mathbf{B}'$ and $\mathbf{B}Cov\left[\boldsymbol{\xi}_{t+1}, \mathbf{z}_t\right]\mathbf{A}'$. Hence we only need to compute $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]$. Recall from above that

$$\tilde{\boldsymbol{\xi}}_{t+1} \equiv \begin{bmatrix} \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \\ \mathbf{0}_{n_x\times 1} \\ \mathbf{v}\left(t+1\right) - \left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right) \\ \mathbf{0}_{n_x\times 1} \\ \left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right) + \left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1} \\ \mathbf{u}_{t+1} - E\left[\mathbf{u}_{t+1}\right] \end{bmatrix}$$

where

$$\mathbf{v}\left(t+1\right) \equiv \left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right) + \left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)$$

and

$$\begin{aligned} \mathbf{u}_{t+1} \equiv\; & \left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right) \\ & + \left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right) + \left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right) \\ & + \left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right) \\ & + \left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right) \end{aligned}$$

Note that

$$E\left[\mathbf{u}_{t+1}\right] = \left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right)E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right]$$

because $\boldsymbol{\epsilon}_{t+1}$ is iid, $E\left[\boldsymbol{\epsilon}_{t+1}\right] = \mathbf{0}$, and $E\left[\mathbf{x}_t^f\right] = \mathbf{0}$. Note also that $E\left[\mathbf{u}_{t+1}\right]$ can be coded directly as:

```
E_eps3 = zeros(ne × ne × ne, 1)
index = 0
for phi1 = 1 : ne
    for phi2 = 1 : ne
        for phi3 = 1 : ne
            index = index + 1
            if phi1 == phi2  &&  phi1 == phi3
                E_eps3(index, 1) = m^3 (ε_{t+1} (phi1))
            end
        end
    end
end
```

Hence,

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right] = E[\tilde{\boldsymbol{\xi}}_{t+1}\left(\tilde{\boldsymbol{\xi}}_{t+1}\right)']$$

$$= E[\begin{bmatrix} \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \\ \mathbf{0}_{n_x\times 1} \\ \mathbf{v}(t+1) - (\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\, vec(\mathbf{I}_{n_e}) \\ \mathbf{0}_{n_x\times 1} \\ (\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s) + \left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1} \\ \mathbf{u}_{t+1} - E[\mathbf{u}_{t+1}] \end{bmatrix}$$
$$\times[\; \sigma\boldsymbol{\epsilon}_{t+1}'\boldsymbol{\eta}' \quad \mathbf{0}_{1\times n_x} \quad \mathbf{v}(t+1)' - vec(\mathbf{I}_{n_e})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})' \quad \mathbf{0}_{1\times n_x}$$
$$(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})' + \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)' + \boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)' \quad \mathbf{u}_{t+1}' - E\left[\mathbf{u}_{t+1}'\right]\;]$$

$$Var\left(\tilde{\boldsymbol{\xi}}_{t+1}\right) = E[\begin{bmatrix} \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\sigma\boldsymbol{\epsilon}_{t+1}'\boldsymbol{\eta}' & \mathbf{0}_{n_x\times n_x} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{13} & \mathbf{0}_{n_x\times n_x} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{15} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{16} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x^3} \\ Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{31} & \mathbf{0}_{n_x^2\times n_x} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{33} & \mathbf{0}_{n_x\times n_x} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{35} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{36} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x^3} \\ Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{51} & \mathbf{0}_{n_x^2\times n_x} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{53} & \mathbf{0}_{n_x\times n_x} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{55} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{56} \\ Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{61} & \mathbf{0}_{n_x^3\times n_x} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{63} & \mathbf{0}_{n_x\times n_x} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{65} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{66} \end{bmatrix}]$$

where we have defined:

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{13} \equiv E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\mathbf{v}(t+1)' - vec(\mathbf{I}_{n_e})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'\right)]$$

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{33} \equiv E[(\mathbf{v}(t+1) - (\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\, vec(\mathbf{I}_{n_e}))\left(\mathbf{v}(t+1)' - vec(\mathbf{I}_{n_e})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'\right)]$$

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{15} \equiv E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left((\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})' + \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)' + \boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right)]$$

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{16} \equiv E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\mathbf{u}_{t+1}' - E\left[\mathbf{u}_{t+1}'\right]\right)]$$

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{35} \equiv E[(\mathbf{v}(t+1) - (\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\, vec(\mathbf{I}_{n_e}))$$
$$\times\left((\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})' + \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)' + \boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right)]$$

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{36} \equiv E[(\mathbf{v}(t+1) - (\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\, vec(\mathbf{I}_{n_e}))\left(\mathbf{u}_{t+1}' - E\left[\mathbf{u}_{t+1}'\right]\right)]$$

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{55} \equiv E[\left((\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s) + \left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\right)$$
$$\times\left((\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})' + \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)' + \boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right)]$$

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{56} \equiv E[\left((\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s) + \left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{u}_{t+1}' - E\left[\mathbf{u}_{t+1}'\right]\right)]$$

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{66} \equiv E[(\mathbf{u}_{t+1} - E[\mathbf{u}_{t+1}])\left(\mathbf{u}_{t+1}' - E\left[\mathbf{u}_{t+1}'\right]\right)]$$

We have already derived the expressions for $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{13}$ and $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{33}$, and we will now compute the remaining terms.

4.3.1 For $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{15}$

Note first that $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{15}$ has dimensions $n_x \times n_x^2$.

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{15} = E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)' + \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)' + \boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right)]$$

$$= E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)' + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)' + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)']$$

$$= E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}'\otimes(\mathbf{x}_t^s)'\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)' + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\right)'\otimes\left(\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)' + \sigma\boldsymbol{\eta}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)']$$

$$= E[\sigma\boldsymbol{\eta}\left(\boldsymbol{\epsilon}_{t+1}\otimes 1\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes(\mathbf{x}_t^s)'\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)' + \sigma\boldsymbol{\eta}\left(\boldsymbol{\epsilon}_{t+1}\otimes 1\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\right)'\otimes\left(\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)' + \sigma\boldsymbol{\eta}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)']$$

$$= E[\sigma\boldsymbol{\eta}\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes(\mathbf{x}_t^s)'\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)' + \sigma\boldsymbol{\eta}\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\right)'\otimes\left(\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)' + \sigma\boldsymbol{\eta}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)']$$

$=$

1) $\quad \sigma\boldsymbol{\eta}\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^s\right]'\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)'$

2) $\quad +\sigma\boldsymbol{\eta}\left(\mathbf{I}_{n_e}\otimes\left(E\left[\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right]\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

3) $\quad +\sigma\boldsymbol{\eta}\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$

Checking the dimensions:
Term 1: $(n_x\times n_e)(n_e\times n_e n_x)(n_x n_x\times n_e n_x)'$ ok
Term 2: $(n_x\times n_e)(n_e\times n_e n_x n_x)(n_x n_x\times n_e n_x n_x)'$ ok
Term 3: $(n_x\times n_e)(n_x n_x\times n_e)'$ ok

4.3.2 For $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{16}$

Note first that $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{16}$ has dimensions $n_x \times n_x^3$.

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{16} = E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\mathbf{u}_{t+1}' - E\left[\mathbf{u}_{t+1}'\right]\right)]$$

$$= E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\mathbf{u}_{t+1}']$$

$$\begin{aligned}= E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}(&\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\right)' \\ &+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)' + \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)' \\ &+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)' + \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\boldsymbol{\sigma\eta}\right)' \\ &+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)' + \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right)')]\end{aligned}$$

$$\begin{aligned}&= \sigma\boldsymbol{\eta}E\left(\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right)\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\right)' \\ &+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]\left(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)' + \sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]\left(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'\end{aligned}$$

$$+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right](\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right](\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right](\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right](\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})']$$

$$=\sigma\boldsymbol{\eta}E\left((1\otimes\boldsymbol{\epsilon}_{t+1})\left(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}_{t+1}'\right)\right)(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right](\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'+\mathbf{0}$$
$$+\sigma\boldsymbol{\eta}E\left[(\boldsymbol{\epsilon}_{t+1}\otimes 1)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\right](\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'+\mathbf{0}$$
$$+\mathbf{0}+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right](\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})']$$

$$=\sigma\boldsymbol{\eta}E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\right](\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right](\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right](\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right](\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})']$$

$$=\sigma\boldsymbol{\eta}\left(E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\otimes\mathbf{I}_{n_e}\right)(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right](\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+\sigma\boldsymbol{\eta}\left(\mathbf{I}_{n_e}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right](\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})']$$

Hence we only need to compute directly the terms $E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$ and $E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$.
We first note that
$E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$

And
$$E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]=E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}'\right)\right]$$
$$=E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left\{\left\{\epsilon_{t+1}(1,\phi_3)\right\}_{\phi_3=1}^{n_e}\epsilon_{t+1}(1,\phi_4)\right\}_{\phi_4=1}^{n_e}\right)\right]$$
$$=E\left[\begin{matrix}\epsilon_{t+1}(\phi_1,1)\\ \epsilon_{t+1}(\phi_2,1)\\ \dots\\ \epsilon_{t+1}(\phi_4,1)\end{matrix}\left(\left\{\epsilon_{t+1}(1,\phi_2)\left\{\left\{\epsilon_{t+1}(1,\phi_3)\right\}_{\phi_3=1}^{n_e}\epsilon_{t+1}(1,\phi_4)\right\}_{\phi_4=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right)\right]$$

Thus, the quasi Matlab codes are
$E\_eps\_eps3 = zeros(ne,(ne)^3)$
$for\ phi1 = 1 : ne$
    $index2 = 0$
    $for\ phi2 = 1 : ne$
        $for\ phi3 = 1 : ne$
            $for\ phi4 = 1 : ne$

```
            % second moments
            if (phi1 == phi2 && phi3 == phi4 && phi1~ = phi4)
                E_eps_eps3(phi1, index2) = 1
            elseif (phi1 == phi3 && phi2 == phi4 && phi1~ = phi2)
                E_eps_eps3(phi1, index2) = 1
            elseif (phi1 == phi4 && phi2 == phi3 && phi1~ = phi2)
                E_eps_eps3(phi1, index2) = 1
            % fourth moments
            elseif(phi1 == phi2 && phi1 == phi3 && phi1 == phi4)
                E_eps_eps3(phi1, index2) = m^4 (ϵ_{t+1} (phi1))
            end
        end
    end
  end
end
```

4.3.3 For $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{35}$

Note first that $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{35}$ has dimensions $n_x^2 \times n_x^2$.

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{35}$$

$$\equiv E[(\mathbf{v}(t+1) - (\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\,vec(\mathbf{I}_{n_e}))\left((\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})' + \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)' + \boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right)]$$

$$= E[\left((\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right) + (\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}) - (\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\,vec(\mathbf{I}_{n_e})\right)$$
$$\times\left((\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})' + \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)' + \boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right)]$$

$$= E[$$
$$(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$
$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$
$$-(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\,vec(\mathbf{I}_{n_e})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$-(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\,vec(\mathbf{I}_{n_e})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$-(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\,vec(\mathbf{I}_{n_e})\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$
$$]$$

$$= E[$$

$(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) (\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$

$+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

$+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\right) \left(\sigma\boldsymbol{\eta} \otimes \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$

$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}' \otimes \mathbf{x}_t^f \left(\mathbf{x}_t^s\right)'\right) (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$

$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}' \otimes \mathbf{x}_t^f \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right) \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}' \otimes \mathbf{x}_t^f\right) \left(\sigma\boldsymbol{\eta} \otimes \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$

$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) (\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$

$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) \left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}' \otimes \boldsymbol{\epsilon}_{t+1}\right) \left(\sigma\boldsymbol{\eta} \otimes \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$

-0

-0

-0

]

$= E[$

$(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) (\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$

$+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

$+0$

$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}' \otimes \mathbf{x}_t^f \left(\mathbf{x}_t^s\right)'\right) (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$

$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}' \otimes \mathbf{x}_t^f \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right) \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

$+0$

$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) (\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$

$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) \left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}' \otimes \boldsymbol{\epsilon}_{t+1}\right) \left(\sigma\boldsymbol{\eta} \otimes \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$

]

$= E[$

$(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) (\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$

$+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\mathbf{I}_{n_e} \otimes \mathbf{x}_t^f \left(\mathbf{x}_t^s\right)'\right) (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$

$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\mathbf{I}_{n_e} \otimes \mathbf{x}_t^f \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right) \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) (\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$

$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) \left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}' \otimes \boldsymbol{\epsilon}_{t+1}\right) \left(\sigma\boldsymbol{\eta} \otimes \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$

]

Note that

$(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

$= (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) ((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) \otimes 1) \left(\boldsymbol{\epsilon}_{t+1}' \otimes \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right) \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

$$= (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) \left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) \boldsymbol{\epsilon}'_{t+1} \otimes \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \right)' \right) \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}} \right)'$$

$$= (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) \left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) (\boldsymbol{\epsilon}'_{t+1} \otimes 1) \otimes \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \right)' \right) \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}} \right)'$$

$$= (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) \left((\boldsymbol{\epsilon}_{t+1} \boldsymbol{\epsilon}'_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) \otimes \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \right)' \right) \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}} \right)'$$

So

$Var\left[\boldsymbol{\xi}_{t+1}\right]_{35} =$

1) $(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s)'\right](\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$

2) $+(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

3) $+(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})\left(\mathbf{I}_{n_e} \otimes E\left[\mathbf{x}_t^f (\mathbf{x}_t^s)'\right]\right)(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$

4) $+(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})\left(\mathbf{I}_{n_e} \otimes E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

5) $+(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) E\left[(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s)'\right](\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$

6) $+(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left(E\left[(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})\right] \otimes E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

7) $+(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left(E\left[\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right]\right)\left(\sigma\boldsymbol{\eta} \otimes \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$

Checking the dimensions:

Term 1: $(n_x n_x \times n_x n_e)(n_x n_e \times n_e n_x)(n_e n_x \times n_x n_x)$ *ok*

Term 2: $(n_x n_x \times n_x n_e)\left(n_x n_e \times n_e n_x^2\right)\left(n_e n_x^2 \times n_x^2\right)$ *ok*

Term 3: $(n_x n_x \times n_e n_x)(n_e n_x \times n_x n_e)(n_e n_x \times n_x n_x)$ *ok*

Term 4: $\left(n_x^2 \times n_e n_x\right)(n_e n_x \times n_e n_x n_x)(n_e n_x n_x \times n_x n_x)$ *ok*

Term 5: $\left(n_x^2 \times n_e^2\right)\left(n_e^2 \times n_e n_x\right)(n_e n_x \times n_x n_x)$ *ok*

Term 6: $\left(n_x^2 \times n_e^2\right)\left(n_e^2 \times n_e n_x^2\right)(n_e n_x n_x \times n_x n_x)$ *ok*

Term 7: $\left(n_x^2 \times n_e^2\right)\left(n_e^2 \times n_e\right)(n_e \times n_x n_x)$ *ok*

We then need to show how to compute the following matrices

$$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s)'\right] = E\left[\left\{x_t^f(\gamma_1,1)\boldsymbol{\epsilon}_{t+1}\right\}_{\gamma_1=1}^{n_x}\left(\left\{\epsilon_{t+1}(\phi_2,1)\mathbf{x}_t^s\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{\epsilon_{t+1}(\phi_1,1)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\left(\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^s(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are

```
E_xfeps_epsxs = zeros(nx ne, ne nx)
index1 = 0
for gama1 = 1 : nx
    for phi1 = 1 : ne
        index1 = index1 + 1
        index2 = 0
        for phi2 = 1 : ne
            for gama2 = 1 : nx
                index2 = index2 + 1
                if phi1 = phi2
                    E_xfeps_epsxs(index1, index2) = E_xf_xs(gama1, gama2)
                end
            end
```

end
end
end
where $E\_xf\_xs = E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^s\right)'\right] = reshape(\left(E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right]\right)', nx, nx)$. This is so because

$$E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right] = E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^s(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right]$$

$$= E\begin{bmatrix} x_t^f(\gamma_1,1)\left\{x_t^s(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x} \\ x_t^f(\gamma_2,1)\left\{x_t^s(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x} \\ \dots \\ x_t^f(n_e,1)\left\{x_t^s(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x} \end{bmatrix}$$

So simply doing (for a 2 by 2 matrix)

$$reshape(E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right], nx, nx) = E\begin{bmatrix} x^f(1,1)\,x^s(1,1) & x^f(2,1)\,x^s(1,1) \\ x^f(1,1)\,x^s(2,1) & x^f(2,1)\,x^s(2,1) \end{bmatrix}$$

and we therefore need to transpose $E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right]$ in the expression above.

And

$$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{\epsilon_{t+1}(\phi_1,1)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\left(\left\{\epsilon_{t+1}(\phi_2,1)\left\{\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x^2}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{\epsilon_{t+1}(\phi_1,1)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\left(\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_2,1)\left\{x_t^f(\gamma_3,1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_xfeps_epsxfxf = zeros(nxne, ne(nx)^2)
index1 = 0
for gama1 = 1 : nx
    for phi1 = 1 : ne
        index1 = index1 + 1
        index2 = 0
        for phi2 = 1 : ne
            for gama2 = 1 : nx
                for gama3 = 1 : nx
                    index2 = index2 + 1
                    if phi1 == phi2
                        E_xfeps_epsxfxf(index1, index2) = E_xf_xf_xf(gama1, gama2, gama3)
                    end
                end
            end
        end
    end
end
```

where $E\_xf\_xf\_xf = reshape(\left(E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right]\right), nx, nx, nx)$

And
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\right]$

$$= E\left[\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\left(\left\{\epsilon_{t+1}(\phi_3,1)\left\{x_t^s(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_3=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_eps2_epsxs = zeros(n_e n_e, n_e n_x)
index1 = 0
for phi1 = 1 : ne
    for phi2 = 1 : ne
        index1 = index1 + 1
        index2 = 0
        for phi3 = 1 : ne
            for gama1 = 1 : nx
                index2 = index2 + 1
                if phi1 = phi2 && phi1 == phi3
                    E_eps2_epsxs (index1, index2) = E_xs(gama1, 1) × m^3 (ε_{t+1} (phi1))
                end
            end
        end
    end
end
```

Finally:
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}\right)\right] = E\left[\left(\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}'(1,\phi_2)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right]$

$$= E\left[\left\{\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}'(1,\phi_2)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\epsilon_{t+1}(\phi_3,1)\right\}_{\phi_3=1}^{n_e}\right]$$

Thus the quasi Matlab codes are:

```
E_eps2_eps = zeros((ne)^2, ne)
for phi2 = 1 : ne
    index1 = 0
    for phi1 = 1 : ne
        for phi3 = 1 : ne
            index1 = index1 + 1
            if phi1 == phi2 && phi1 == phi3
                E_eps2_eps(index1, phi2) = m^3 (ε_{t+1} (phi1))
            end
        end
    end
end
```

4.3.4 For $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{36}$

Note first that $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{36}$ has dimensions $n_x^2\times n_x^3$.

$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{36}\equiv E[(\mathbf{v}(t+1)-(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\,vec(\mathbf{I}_{n_e}))\left(\mathbf{u}_{t+1}'-E\left[\mathbf{u}_{t+1}'\right]\right)]$

$$
\begin{aligned}
&= E[\Big((\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) + (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) - (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\, vec\left(\mathbf{I}_{n_e}\right) \Big) \\
&\qquad \times \left(\mathbf{u}'_{t+1} - E\left[\mathbf{u}'_{t+1}\right]\right)]
\end{aligned}
$$

$$
\begin{aligned}
&= E[\Big((\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) + (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) - (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\, vec\left(\mathbf{I}_{n_e}\right) \Big) \mathbf{u}'_{t+1}] \\
{}' \quad & -E[\Big((\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) + (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) - (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\, vec\left(\mathbf{I}_{n_e}\right) \Big) E\left[\mathbf{u}'_{t+1}\right]]
\end{aligned}
$$

$$
\begin{aligned}
&= E[\Big((\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) + (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) - (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\, vec\left(\mathbf{I}_{n_e}\right) \Big) \mathbf{u}'_{t+1}] \\
{}' \quad & -E[(0+0+0)\, E\left[\mathbf{u}'_{t+1}\right]]
\end{aligned}
$$

$$
\begin{aligned}
&= E[(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) \mathbf{u}'_{t+1} \\
&\qquad + (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) \mathbf{u}'_{t+1} \\
&\qquad + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) \mathbf{u}'_{t+1} \\
&\qquad - (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\, vec\left(\mathbf{I}_{n_e}\right) \mathbf{u}'_{t+1}]
\end{aligned}
$$

$$
\begin{aligned}
&= E[(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) \mathbf{u}'_{t+1} \\
&\qquad + (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) \mathbf{u}'_{t+1} \\
&\qquad + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) \mathbf{u}'_{t+1}] \\
&\qquad - (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\, vec\left(\mathbf{I}_{n_e}\right) E\left[\mathbf{u}'_{t+1}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= E[\\
&\quad (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) (\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})' \\
&\qquad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})' \\
&\qquad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})' + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})' \\
&\qquad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})' + (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})') \\
\\
&\quad (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) (\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})' \\
&\qquad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})' \\
&\qquad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})' + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})' \\
&\qquad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})' + (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})') \\
\\
&\quad (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) (\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})' \\
&\qquad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})' \\
&\qquad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})' + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})' \\
&\qquad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})' + (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})')] \\
&- (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\, vec\left(\mathbf{I}_{n_e}\right) E\left[\mathbf{u}'_{t+1}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= E[\\
&\quad (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) (\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'
\end{aligned}
$$

$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x})'+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x}\otimes\mathbf{h}_\mathbf{x})'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h}_\mathbf{x})'+0)$$

$$(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_\mathbf{x}\otimes\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x})'+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x}\otimes\mathbf{h}_\mathbf{x})'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h}_\mathbf{x})'+0)$$

$$(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_\mathbf{x}\otimes\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x})'+0$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x}\otimes\mathbf{h}_\mathbf{x})'+0$$
$$+0+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')]$$

$$-(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\,vec\left(\mathbf{I}_{n_e}\right)E\left[\mathbf{u}_{t+1}'\right]$$

$$=E[$$
$$(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}_{t+1}'\right)(\mathbf{h}_\mathbf{x}\otimes\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta})'$$
$$+(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x})'$$
$$+(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\left(\mathbf{x}_t^f\right)'\otimes(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'\right)(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x}\otimes\mathbf{h}_\mathbf{x})'$$
$$+(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x}\otimes\boldsymbol{\sigma\eta})'$$
$$+(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left((\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'\otimes\left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h}_\mathbf{x})'$$

$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_\mathbf{x}\otimes\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x}\otimes\mathbf{h}_\mathbf{x})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x}\otimes\boldsymbol{\sigma\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left((\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'\otimes\left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h}_\mathbf{x})'$$

$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_\mathbf{x}\otimes\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x})'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right)']$$

$$-\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right)E\left[\mathbf{u}_{t+1}'\right]$$

$$=E[$$

$$\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}_{t+1}'\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'$$
$$+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$$
$$+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right)\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'$$
$$+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)'$$
$$+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta}\right)'$$
$$+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\otimes\left(\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x}\right)'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes\mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\otimes\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x}\right)'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(1\otimes\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right)\left(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}_{t+1}'\right)\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\otimes1\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}'\right)\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right)']$$

$$-\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right)E\left[\mathbf{u}_{t+1}'\right]$$

$$=$$

1) $$\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\otimes\mathbf{I}_{n_e}\right)\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'$$

2) $$+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$$

3) $$+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\otimes E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]\right)\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'$$

4) $$+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\right]\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)'$$

5) $$+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta}\right)'$$

6) $\quad + (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left((\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})' \otimes \left(\mathbf{x}_t^f\right)'\right)\right](\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'$

7) $\quad + (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right](\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$

8) $\quad + (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right](\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$

9) $\quad + (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right](\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$

10) $\quad + (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})\left(\mathbf{I}_{n_e} \otimes E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$

11) $\quad + (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right](\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$

12) $\quad + (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})\left(E\left[\boldsymbol{\epsilon}_{t+1}(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'\right] \otimes E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\right)(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'$

13) $\quad + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left(E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right] \otimes E\left[(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\boldsymbol{\epsilon}_{t+1}'\right]\right)(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$

14) $\quad + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) E\left[(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right](\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$

15) $\quad + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left(E\left[(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\boldsymbol{\epsilon}_{t+1}'\right] \otimes E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$

16 $\quad + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) E\left[(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\boldsymbol{\epsilon}_{t+1}' \otimes \left(\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}'\right)\right)\right](\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})'$

17) $\quad - (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\, vec(\mathbf{I}_{n_e}) E\left[\mathbf{u}_{t+1}'\right]$

Hence, we need to compute the remaining matrices directly. This is done below where the number relates to the row in the expression for $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{36}$

1) None

2)

$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$

$$= E\left[\left(\left\{x_t^f(\gamma_1,1)\left\{\epsilon_{t+1}(\phi_1,1)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)\left(\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_3,1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

$E\_xfeps\_xfepsxf = zeros(nx \times ne, nx \times ne \times nx)$
$index1 = 0$
$for\ gama1 = 1 : nx$
$\quad for\ phi1 = 1 : ne$
$\quad\quad index1 = index1 + 1$
$\quad\quad index2 = 0$
$\quad\quad for\ gama2 = 1 : nx$
$\quad\quad\quad for\ phi2 = 1 : ne$
$\quad\quad\quad\quad for\ gama3 = 1 : nx$
$\quad\quad\quad\quad\quad index2 = index2 + 1$
$\quad\quad\quad\quad\quad if\ phi1 == phi2$
$\quad\quad\quad\quad\quad\quad E\_xfeps\_xfepsxf(index1, index2) = E\_xf\_xf\_xf(gama1, gama2, gama3)$

end
end
end
end
end
end
where $E\_xf\_xf\_xf = reshape(E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right], nx, nx, nx)$

3)
$E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right] = E\left[\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1} \otimes \boldsymbol{\epsilon}'_{t+1}\right] = \left(E\left[\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\right]\right)'$
but $E\left[\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\right]$ is already computed

4)
$$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}'_{t+1} \otimes \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right)\right]$$

$$= E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$$

$$= E\left[\left(\left\{x_t^f(\gamma_1,1)\left\{\epsilon_{t+1}(\phi_1,1)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)\left(\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_2,1)\left\{x_t^f(\gamma_3,1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are:
$E\_xfeps\_epsxfxf = zeros(nx \times ne, ne \times nx \times nx)$
$index1 = 0$
$for\ gama1 = 1 : nx$
$for\ phi1 = 1 : ne$
$index1 = index1 + 1$
$index2 = 0$
$for\ phi2 = 1 : ne$
$for\ gama2 = 1 : nx$
$for\ gama3 = 1 : nx$
$index2 = index2 + 1$
$if\ phi1 == phi2$
$E\_xfeps\_epsxfxf(index1, index2) = E\_xf\_xf\_xf(gama1, gama2, gama3)$
end
end
end
end
end
end
where $E\_xf\_xf\_xf = reshape(E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right], nx, nx, nx)$

5)
$$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left(\left\{x_t^f(\gamma_1,1)\left\{\epsilon_{t+1}(\phi_1,1)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)\left(\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_3,1)\right\}_{\phi_3=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_xfeps_epsxfeps = zeros(nx × ne, ne × nx × ne)
index1 = 0
for gama1 = 1 : nx
    for phi1 = 1 : ne
        index1 = index1 + 1
        index2 = 0
        for phi2 = 1 : ne
            for gama2 = 1 : nx
                for phi3 = 1 : ne
                    index2 = index2 + 1
                    if phi1 == phi2 && phi1 == phi3
                        E_xfeps_epsxfeps(index1, index2) = E_xf_xf(gama1, gama2) × m^3 (ϵ_{t+1} (phi1))
                    end
                end
            end
        end
    end
end
```

where $E\_xf\_xf = reshape(E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right], nx, nx)$

6)

$$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)' \otimes \left(\mathbf{x}_t^f\right)'\right)\right]$$

$$= E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$$

$$= E\left[\left(\left\{x_t^f\left(\gamma_1,1\right)\left\{\epsilon_{t+1}\left(\phi_1,1\right)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)\left(\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{\epsilon_{t+1}\left(\phi_3,1\right)\left\{x_t^f\left(\gamma_2,1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_xfeps_eps2xf = zeros(nx × ne, ne × ne × nx)
index1 = 0
for gama1 = 1 : nx
    for phi1 = 1 : ne
        index1 = index1 + 1
        index2 = 0
        for phi2 = 1 : ne
            for phi3 = 1 : ne
                for gama2 = 1 : nx
                    index2 = index2 + 1
                    if phi1 == phi2 && phi1 == phi3
                        E_xfeps_eps2xf(index1, index2) = E_xf_xf(gama1, gama2) × m^3 (ϵ_{t+1} (phi1))
                    end
                end
            end
        end
    end
end
```

where $E\_xf\_xf = reshape(E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right], nx, nx)$

7)

$$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$=E\left[\left(\left\{\epsilon_{t+1}\left(\phi_1,1\right)\left\{x_t^f\left(\gamma_1,1\right)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{x_t^f\left(\gamma_2,1\right)\left\{x_t^f\left(\gamma_3,1\right)\left\{\epsilon_{t+1}\left(\phi_2,1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_epsxf_xfxfeps = zeros(ne × nx, nx × nx × ne)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for gama2 = 1 : nx
            for gama3 = 1 : nx
                for phi2 = 1 : ne
                    index2 = index2 + 1
                    if phi1 == phi2
                        E_epsxf_xfxfeps(index1, index2) = E_xf_xf_xf(gama1, gama2, gama3)
                    end
                end
            end
        end
    end
end
```

8)

$$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$$

$$=E\left[\left(\left\{\epsilon_{t+1}\left(\phi_1,1\right)\left\{x_t^f\left(\gamma_1,1\right)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{x_t^f\left(\gamma_2,1\right)\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{x_t^f\left(\gamma_3,1\right)\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_epsxf_xfepsxf = zeros(ne × nx, nx × ne × nx)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for gama2 = 1 : nx
            for phi2 = 1 : ne
                for gama3 = 1 : nx
                    index2 = index2 + 1
                    if phi1 == phi2
                        E_epsxf_xfepsxf(index1, index2) = E_xf_xf_xf(gama1, gama2, gama3)
                    end
                end
            end
        end
    end
```

end

9)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$=E\left[\left(\left\{\epsilon_{t+1}\left(\phi_1,1\right)\left\{x_t^f\left(\gamma_1,1\right)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{x_t^f\left(\gamma_2,1\right)\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{\epsilon_{t+1}\left(\phi_3,1\right)\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_epsxf_xfeps2 = zeros(ne × nx, nx × ne × ne)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for gama2 = 1 : nx
            for phi2 = 1 : ne
                for phi3 = 1 : ne
                    index2 = index2 + 1
                    if phi1 == phi2 && phi1 == phi3
                        E_epsxf_xfeps2(index1, index2) = E_xf_xf(gama1, gama2) × m^3 (ε_{t+1} (phi1))
                    end
                end
            end
        end
    end
end
```

10)
$\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)$

where $E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]=reshape(E\left[\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right],nx,(nx)^2)$

11)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$=E\left[\left(\left\{\epsilon_{t+1}\left(\phi_1,1\right)\left\{x_t^f\left(\gamma_1,1\right)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{x_t^f\left(\gamma_2,1\right)\left\{\epsilon_{t+1}\left(\phi_3,1\right)\right\}_{\phi_3=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_epsxf_epsxfeps = zeros(ne × nx, ne × nx × ne)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for phi2 = 1 : ne
            for gama2 = 1 : nx
                for phi3 = 1 : ne
```

$index2 = index2 + 1$
$if\ phi1 == phi2\ \&\&\ phi1 == phi3$
$E\_epsxf\_xfeps2(index1, index2) = E\_xf\_xf(gama1, gama2) \times m^3\left(\epsilon_{t+1}\left(phi1\right)\right)$
end
end
end
end
end
end

12) none

13)
$\left(E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right] \otimes E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}_{t+1}'\right]\right)$
Here we only need to compute
$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}_{t+1}'\right] = E\left[\left\{\epsilon_{t+1}\left(\phi_1, 1\right)\left\{\epsilon_{t+1}\left(\phi_2, 1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\left\{\epsilon_{t+1}'\left(1, \phi_3\right)\right\}_{\phi_3=1}^{n_e}\right]$
Thus the quasi Matlab codes are:
$E\_eps2\_eps = zeros(ne \times ne, ne)$
$index1 = 0$
$for\ phi1 = 1 : ne$
$for\ phi2 = 1 : ne$
$index1 = index1 + 1$
$index2 = 0$
$for\ phi3 = 1 : ne$
$index2 = index2 + 1$
$if\ phi1 == phi2\ \&\&\ phi1 == phi3$
$E\_eps2\_eps(index1, index2) = m^3\left(\epsilon_{t+1}\left(phi1\right)\right)$
end
end
end
end
end
But we already know E_eps2_eps from previous derivations.
14)
$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$

$$= E\left[\left(\left\{\epsilon_{t+1}\left(\phi_1, 1\right)\left\{\epsilon_{t+1}\left(\phi_2, 1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{x_t^f\left(\gamma_1, 1\right)\left\{\epsilon_{t+1}\left(\phi_3, 1\right)\left\{x_t^f\left(\gamma_2, 1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_3=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:
$E\_eps2\_xfepsxf = zeros(ne \times ne, nx \times ne \times nx)$
$index1 = 0$
$for\ phi1 = 1 : ne$
$for\ phi2 = 1 : ne$
$index1 = index1 + 1$
$index2 = 0$
$for\ gama1 = 1 : nx$
$for\ phi3 = 1 : ne$

$for\ gama2 = 1 : nx$

$index2 = index2 + 1$

$if\ phi1 == phi2\ \&\&\ phi1 == phi3$

$E\_eps2\_xfepsxf(index1, index2) = E\_xf\_xf(gama1, gama2) \times m^3\left(\epsilon_{t+1}\left(phi1\right)\right)$

end

end

end

end

end

end

15)
$\left(E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}'_{t+1}\right] \otimes E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\right)$

None since we know $E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}'_{t+1}\right]$

16)
$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}'_{t+1} \otimes \left(\boldsymbol{\epsilon}'_{t+1} \otimes \boldsymbol{\epsilon}'_{t+1}\right)\right)\right]$

$$= E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left(\left\{\epsilon_{t+1}\left(\phi_1, 1\right)\left\{\epsilon_{t+1}\left(\phi_2, 1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{\epsilon_{t+1}\left(\phi_3, 1\right)\left\{\epsilon_{t+1}\left(\phi_4, 1\right)\left\{\epsilon_{t+1}\left(\phi_5, 1\right)\right\}_{\phi_5=1}^{n_e}\right\}_{\phi_4=1}^{n_e}\right\}_{\phi_3=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_eps2_eps3 = zeros(ne*ne,ne*ne*ne);
index1 = 0;
    for phi1=1:ne
        for phi2=1:ne
        index1 = index1 + 1;
        index2 = 0;
        for phi3=1:ne
            for phi4=1:ne
                for phi5=1:ne
                    index2 = index2 + 1;
                    % Second order moments times third order moments
                    if phi1 == phi2 & & phi2 == phi3 & & phi4 == phi5 & & phi1 ~= phi4
                    E_eps2_eps3(index1,index2) = vectorMom3(1,phi1);
                    elseif phi1 == phi3 & & phi3 == phi4 & & phi2 == phi5 & & phi1 ~= phi2
                    E_eps2_eps3(index1,index2) = vectorMom3(1,phi1);
                    elseif phi1 == phi4 & & phi4 == phi5 & & phi2 == phi3 & & phi1 ~= phi2
                    E_eps2_eps3(index1,index2) = vectorMom3(1,phi1);
                    elseif phi3 == phi4 & & phi4 == phi5 & & phi1 == phi2 & & phi1 ~= phi3
                    E_eps2_eps3(index1,index2) = vectorMom3(1,phi3);
                    elseif phi2 == phi3 & & phi3 == phi4 & & phi1 == phi5 & & phi1 ~= phi2
                    E_eps2_eps3(index1,index2) = vectorMom3(1,phi2);
                    elseif phi1 == phi3 & & phi3 == phi5 & & phi2 == phi4 & & phi1 ~= phi2
                    E_eps2_eps3(index1,index2) = vectorMom3(1,phi1);
                    elseif phi1 == phi2 & & phi1 == phi4 & & phi3 == phi5 & & phi1 ~= phi3
                    E_eps2_eps3(index1,index2) = vectorMom3(1,phi1);
                    elseif phi1 == phi2 & & phi1 == phi5 & & phi3 == phi4 & & phi1 ~= phi3
                    E_eps2_eps3(index1,index2) = vectorMom3(1,phi1);
```

```
                elseif phi2 == phi4 & & phi2 == phi5 & & phi1 == phi3 & & phi1 ~= phi2
                E_eps2_eps3(index1,index2) = vectorMom3(1,phi2);
                elseif phi2 == phi3 & & phi2 == phi5 & & phi1 == phi4 & & phi1 ~= phi2
                E_eps2_eps3(index1,index2) = vectorMom3(1,phi2);
                % Fifth order moments
                elseif phi1 == phi2 & & phi2 == phi3 & & phi3 == phi4 & & phi4 == phi5
                    E_eps2_eps3(index1,index2) = vectorMom5(1,phi1);
                end
            end
        end
    end
end
end
```

17) none

4.3.5 For $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{55}$

Note first that $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{55}$ has dimensions $n_x^2 \times n_x^2$.

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{55} \equiv E[\left(\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\right)$$
$$\times\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'+\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right)]$$

$$= E[$$
$$\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'+\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right)$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'+\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right)$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'+\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right)$$
$$]$$

$$= E[$$
$$\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

]

$= E[$

$$(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes(\mathbf{x}_t^s)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes 1\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes(\mathbf{x}_t^s)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes 1\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes 1\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes(\mathbf{x}_t^s)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes 1\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

]

$= E[$

$$(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes\mathbf{x}_t^s(\mathbf{x}_t^s)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes\mathbf{x}_t^s\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)(\mathbf{x}_t^s)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes(\mathbf{x}_t^s)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

]

$$=(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^s(\mathbf{x}_t^s)'\right]\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^s\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\mathbf{I}_{n_e}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)(\mathbf{x}_t^s)'\right]\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\mathbf{I}_{n_e}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\mathbf{I}_{n_e}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\mathbf{I}_{n_e}\otimes E\left[(\mathbf{x}_t^s)'\right]\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\mathbf{I}_{n_e}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

$=$

1) $(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^s(\mathbf{x}_t^s)'\right]\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'$

2) $+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

3) $+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^s\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)')$

4) $+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\mathbf{I}_{n_e}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)(\mathbf{x}_t^s)'\right]\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'$

5) $+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\mathbf{I}_{n_e}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

6) $+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\mathbf{I}_{n_e}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$

7) $+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\mathbf{I}_{n_e}\otimes E\left[(\mathbf{x}_t^s)'\right]\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'$

8) $+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\mathbf{I}_{n_e}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

9) $+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$

Note here that we already know $E\left[\mathbf{x}_t^s(\mathbf{x}_t^s)'\right]$, $E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]$ and $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]$ from the variance of the states using a second order approximation. This is because

$$Var\left[\mathbf{x}_t^s\right]=E\left[\left(\mathbf{x}_t^s-E\left[\mathbf{x}_t^s\right]\right)\left(\mathbf{x}_t^s-E\left[\mathbf{x}_t^s\right]\right)'\right]$$
$$=E\left[\left(\mathbf{x}_t^s-E\left[\mathbf{x}_t^s\right]\right)\left((\mathbf{x}_t^s)'-E\left[\mathbf{x}_t^s\right]'\right)\right]$$
$$=E\left[\mathbf{x}_t^s(\mathbf{x}_t^s)'-\mathbf{x}_t^sE\left[\mathbf{x}_t^s\right]'-E\left[\mathbf{x}_t^s\right](\mathbf{x}_t^s)'+E\left[\mathbf{x}_t^s\right]E\left[\mathbf{x}_t^s\right]'\right]$$
$$=E\left[\mathbf{x}_t^s(\mathbf{x}_t^s)'\right]-E\left[\mathbf{x}_t^s\right]E\left[\mathbf{x}_t^s\right]'$$
$$\Updownarrow$$
$$Var\left[\mathbf{x}_t^s\right]+E\left[\mathbf{x}_t^s\right]E\left[\mathbf{x}_t^s\right]'=E\left[\mathbf{x}_t^s(\mathbf{x}_t^s)'\right]$$

and

$$Var\left(\mathbf{x}_t^s,\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right)=E\left[\left(\mathbf{x}_t^s-E\left[\mathbf{x}_t^s\right]\right)\left(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)-E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right]\right)'\right]$$
$$=E\left[\left(\mathbf{x}_t^s-E\left[\mathbf{x}_t^s\right]\right)\left(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'-E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)\right]$$
$$=E[\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'-\mathbf{x}_t^sE\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]-E\left[\mathbf{x}_t^s\right]\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'+E\left[\mathbf{x}_t^s\right]E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]]$$
$$=E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]-E\left[\mathbf{x}_t^s\right]E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]$$
$$\Updownarrow$$
$$Var\left(\mathbf{x}_t^s,\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right)+E\left[\mathbf{x}_t^s\right]E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]=E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]$$

and

$$Var\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] = E\left[\left(\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) - E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right]\right)\left(\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) - E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right]\right)'\right]$$

$$= E\left[\left(\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) - E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right]\right)\left(\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' - E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\right)\right]$$

$$= E[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' - \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$$
$$-E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right]\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' + E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]]$$

$$= E[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'] - E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$$
$$-E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right] + E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$$

$$= E[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'] - E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$$
$$\Updownarrow$$
$$Var\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] + E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right] = E[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)']$$

4.3.6 For $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{56}$

Note first that $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{56}$ has dimensions $n_x^2 \times n_x^3$.

$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{56}$

$$= E[\left((\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s) + \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{u}_{t+1}' - E\left[\mathbf{u}_{t+1}'\right]\right)]$$

$$= E[\left((\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s) + \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\right)\mathbf{u}_{t+1}'$$
$$-\left((\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s) + \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\right)E\left[\mathbf{u}_{t+1}'\right]]$$

$$= E[(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s)\mathbf{u}_{t+1}' + \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\mathbf{u}_{t+1}' + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\mathbf{u}_{t+1}'$$
$$-(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s)E\left[\mathbf{u}_{t+1}'\right] - \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)E\left[\mathbf{u}_{t+1}'\right] - \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}E\left[\mathbf{u}_{t+1}'\right]]$$

$$= E[(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s)\mathbf{u}_{t+1}' + \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\mathbf{u}_{t+1}' + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\mathbf{u}_{t+1}']$$

$$= E[(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s)\mathbf{u}_{t+1}'$$
$$+\left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\mathbf{u}_{t+1}'$$
$$+\left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\mathbf{u}_{t+1}']$$

$$= E[(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s)(\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'$$
$$+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'$$

$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h}_{\mathbf{x}})'+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h}_{\mathbf{x}})'+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h}_{\mathbf{x}})'+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')$$
]

using the definition of $\mathbf{u}_{t+1}$ where

$$\mathbf{u}'_{t+1}\equiv\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h}_{\mathbf{x}})'+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})'$$

$$=E[(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\boldsymbol{\sigma\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\boldsymbol{\sigma\eta})'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')$$
$$]$$

$$=E[(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\boldsymbol{\sigma\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left((\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'\otimes\left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left((\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'\otimes\left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)(1\otimes\boldsymbol{\epsilon}_{t+1})\left(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}'_{t+1}\right)(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+0$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)(\boldsymbol{\epsilon}_{t+1}\otimes 1)\left(\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})'$$

$$+0$$
$$+0$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right)')$$
]

$$= E[(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\otimes\left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\otimes 1\right)(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\otimes\left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\otimes 1\right)(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\otimes\mathbf{I}_{n_e}\right)(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\mathbf{I}_{n_e}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')$$
]

1) $$= E[(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$

2) $$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$

3) $$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$

4) $$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\mathbf{I}_{n_e}\otimes\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$

5) $$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$

6) $$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\otimes\mathbf{x}_t^s\left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'$$

7) $+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' \otimes \mathbf{x}_t^s\right) (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})'$

8) $+ \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'$

9) $+ \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$

10) $+ \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'$

11) $+ \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right) \left(\mathbf{I}_{n_e} \otimes \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right) (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})'$

12) $+ \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})'$

13) $+ \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right) \left(\boldsymbol{\epsilon}_{t+1} (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' \otimes \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) \left(\mathbf{x}_t^f\right)'\right) (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})'$

14) $+ \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right) \left(\boldsymbol{\epsilon}_{t+1} (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' \otimes \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right) (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})$

15) $+ (\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2) \left(E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right] \otimes \mathbf{I}_{n_e}\right) (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'$

16) $+ (\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2) \boldsymbol{\epsilon}_{t+1} \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$

17) $+ (\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2) \left(\mathbf{I}_{n_e} \otimes E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\right) (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})'$

18) $+ (\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2) \boldsymbol{\epsilon}_{t+1} (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})')$

]

We thus need to explain how to compute each of these terms

1)

$$E\left[(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left(\left\{\epsilon_{t+1}(\phi_1, 1)\left\{x_t^s(\gamma_1, 1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{x_t^f(\gamma_2, 1)\left\{x_t^f(\gamma_3, 1)\left\{\epsilon_{t+1}(\phi_2, 1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_epsxs_xfxfeps = zeros(ne × nx, nx × nx × ne)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for gama2 = 1 : nx
            for gama3 = 1 : nx
                for phi2 = 1 : ne
                    index2 = index2 + 1
                    if phi1 == phi2
                        E_epsxs_xfxfeps(index1, index2) = E_xs_xf_xf(gama1, gama2, gama3)
                    end
                end
            end
        end
```

end
end
where $E\_xs\_xf\_xf = reshape(E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right], nx, nx, nx)$

2)
$E\left[(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$

$$= E\left[\left(\left\{\epsilon_{t+1}(\phi_1, 1)\{x_t^s(\gamma_1, 1)\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{x_t^f(\gamma_2, 1)\left\{\epsilon_{t+1}(\phi_2, 1)\left\{x_t^f(\gamma_3, 1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_epsxs_xfepsxf = zeros(ne × nx, nx × ne × nx)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for gama2 = 1 : nx
            for phi2 = 1 : ne
                for gama3 = 1 : nx
                    index2 = index2 + 1
                    if phi1 == phi2
                        E_epsxs_xfepsxf(index1, index2) = E_xs_xf_xf(gama1, gama2, gama3)
                    end
                end
            end
        end
    end
end
```

3)
$E\left[(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$= E\left[\left(\left\{\epsilon_{t+1}(\phi_1, 1)\{x_t^s(\gamma_1, 1)\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{x_t^f(\gamma_2, 1)\left\{\epsilon_{t+1}(\phi_2, 1)\{\epsilon_{t+1}(\phi_3, 1)\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_epsxs_xfeps2 = zeros(ne × nx, nx × ne × ne)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for gama2 = 1 : nx
            for phi2 = 1 : ne
                for phi3 = 1 : ne
                    index2 = index2 + 1
                    if phi1 == phi2 && phi1 == phi3
                        E_epsxs_xfeps2(index1, index2) = E_xs_xf(gama1, gama2) × m^3 (ε_{t+1} (phi1))
```

end
end
end
end
end
end

4)
None

5)
$E\left[(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$=E\left[\left(\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^s(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_3,1)\right\}_{\phi_3=1}^{n_e}\right\}_{\gamma_2=1}^{n_e}\right\}_{\phi_2=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_epsxs_epsxfeps = zeros(ne × nx, ne × nx × ne)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for phi2 = 1 : ne
            for gama2 = 1 : nx
                for phi3 = 1 : ne
                    index2 = index2 + 1
                    if phi1 == phi2 && phi1 == phi3
                        E_epsxs_epsxfeps(index1, index2) = E_xs_xf(gama1, gama2) × m^3 (ε_{t+1} (phi1))
                    end
                end
            end
        end
    end
end
```

6)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'\otimes\mathbf{x}_t^s\left(\mathbf{x}_t^f\right)'\right)\right]=E\left[\boldsymbol{\epsilon}_{t+1}(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'\right]\otimes E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\right)'\right]$
Note that we already know $E\left[\boldsymbol{\epsilon}_{t+1}(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'\right]$

7)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'\otimes\mathbf{x}_t^s\right)\right]=E\left[\boldsymbol{\epsilon}_{t+1}(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'\right]\otimes E\left[\mathbf{x}_t^s\right]$
Note that we already know $E\left[\boldsymbol{\epsilon}_{t+1}(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'\right]$

8)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$=E\left[\left(\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{x_t^f(\gamma_3,1)\left\{x_t^f(\gamma_4,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_4=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_epsxfxf_xfxfeps = zeros(ne × nx × nx, nx × nx × ne)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        for gama2 = 1 : nx
            index1 = index1 + 1
            index2 = 0
            for gama3 = 1 : nx
                for gama4 = 1 : nx
                    for phi2 = 1 : ne
                        index2 = index2 + 1
                        if phi1 == phi2
                            E_epsxfxf_xfxfeps(index1, index2) = E_xf_xf_xf_xf(gama1, gama2, gama3, gama4)
                        end
                    end
                end
            end
        end
    end
end
```

where $E\_xf\_xf\_xf\_xf = reshape(E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right], nx, nx, nx, nx)$

9)

$$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$$

$$= E\left[\left(\left(\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_4,1)\right\}_{\gamma_4=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right)\right.$$

Thus the quasi Matlab codes are:

```
E_epsxfxf_xfepsxf = zeros(ne × nx × nx, nx × ne × nx)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        for gama2 = 1 : nx
            index1 = index1 + 1
            index2 = 0
            for gama3 = 1 : nx
                for phi2 = 1 : ne
                    for gama4 = 1 : nx
                        index2 = index2 + 1
                        if phi1 == phi2
                            E_epsxfxf_xfepsxf(index1, index2) = E_xf_xf_xf_xf(gama1, gama2, gama3, gama4)
                        end
                    end
                end
            end
        end
    end
end
```

10)

$$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left(\left(\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{\epsilon_{t+1}(\phi_3,1)\right\}_{\phi_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right)\right.\right.$$

Thus the quasi Matlab codes are:

$E\_epsxfxf\_xfeps2 = zeros(ne\times nx\times nx, nx\times ne\times ne)$
$index1 = 0$
$for\ phi1 = 1:ne$
 $for\ gama1 = 1:nx$
  $for\ gama2 = 1:nx$
   $index1 = index1+1$
   $index2 = 0$
   $for\ gama3 = 1:nx$
    $for\ phi2 = 1:ne$
     $for\ phi3 = 1:ne$
      $index2 = index2+1$
      $if\ phi1 == phi2\ \&\&\ \ phi1 == phi3$
       $E\_epsxfxf\_xfeps2(index1,index2) = E\_xf\_xf\_xf(gama1,gama2,gama3)\times m^3\left(\epsilon_{t+1}(ph\right.$
      end
     end
    end
   end
  end
 end
end

11)

None as $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]$ is known

12)

$$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left(\left(\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_3,1)\right\}_{\phi_3=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)\right.\right.$$

Thus the quasi Matlab codes are:

$E\_epsxfxf\_epsxfeps = zeros(ne\times nx\times nx, ne\times nx\times ne)$
$index1 = 0$
$for\ phi1 = 1:ne$
 $for\ gama1 = 1:nx$
  $for\ gama2 = 1:nx$
   $index1 = index1+1$
   $index2 = 0$
   $for\ phi2 = 1:ne$
    $for\ gama3 = 1:nx$
     $for\ phi3 = 1:ne$
      $index2 = index2+1$

$if\ phi1 == phi2\ \&\&\ phi1 == phi3$
$E\_epsxfxf\_epsxfeps(index1, index2)$
$= E\_xf\_xf\_xf(gama1, gama2, gama3) \times m^3 \left(\epsilon_{t+1}\left(phi1\right)\right)$
end
end
end
end
end
end
end

13)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right)\right] = E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right]$
Note that we already know $E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$ and $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right] = reshape(E\left[\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right], (nx)^2, nx)$

14)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right)\right] = E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]\otimes E\left[\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right]$
Note that we already know $E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

15) None

16)
$E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$

$$= E\left[\left(\left\{\epsilon_{t+1}\left(\phi_1, 1\right)\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{x_t^f\left(\gamma_1, 1\right)\left\{\epsilon_{t+1}\left(\phi_2, 1\right)\left\{x_t^f\left(\gamma_2, 1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:
$E\_eps\_xfepsxf = zeros(ne, nx \times ne \times nx)$
$index1 = 0$
$for\ phi1 = 1 : ne$
$index1 = index1 + 1$
$index2 = 0$
$for\ gama1 = 1 : nx$
$for\ phi2 = 1 : ne$
$for\ gama2 = 1 : nx$
$index2 = index2 + 1$
$if\ phi1 == phi2$
$E\_eps\_xfepsxf(index1, index2) = E\_xf\_xf(gama1, gama2)$
end
end
end
end
end

17)
None

18)
None

4.3.7 For $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{66}$

Note first that $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{66}$ has dimensions $n_x^3 \times n_x^3$.

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{66} = E[(\mathbf{u}_{t+1} - E\left[\mathbf{u}_{t+1}\right])\left(\mathbf{u}'_{t+1} - E\left[\mathbf{u}'_{t+1}\right]\right)]$$

$$= E[\mathbf{u}_{t+1}\mathbf{u}'_{t+1} - \mathbf{u}_{t+1}E\left[\mathbf{u}'_{t+1}\right] - E\left[\mathbf{u}_{t+1}\right]\mathbf{u}'_{t+1} + E\left[\mathbf{u}_{t+1}\right]E\left[\mathbf{u}'_{t+1}\right]]$$

$$= E\left[\mathbf{u}_{t+1}\mathbf{u}'_{t+1}\right] - E\left[\mathbf{u}_{t+1}\right]E\left[\mathbf{u}'_{t+1}\right]$$

We already know $E\left[\mathbf{u}_{t+1}\right]$ so we only need to compute the first term. Hence

$$E\left[\mathbf{u}_{t+1}\mathbf{u}'_{t+1}\right] = E[$$

$$((\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)$$
$$+(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)$$
$$+(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)$$
$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)$$
$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}))$$
$$(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'$$
$$+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')$$
$$]$$

$$= E[$$

$$(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)$$
$$(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'$$

$$
\begin{aligned}
&+ (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})')
\end{aligned}
$$

$$
\begin{aligned}
+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) \\
&(\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})' \\
&+ \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' \\
&+ \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})' \\
&+ \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})' \\
&+ \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})' \\
&+ \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})' \\
&+ (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})')
\end{aligned}
$$

$$
\begin{aligned}
+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right) \\
&(\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})' \\
&+ \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' \\
&+ \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})' \\
&+ \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})' \\
&+ \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})' \\
&+ \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})' \\
&+ (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})')
\end{aligned}
$$

$$
\begin{aligned}
+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) \\
&(\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})' \\
&+ \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' \\
&+ \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})' \\
&+ \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})' \\
&+ \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})' \\
&+ \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})' \\
&+ (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})')
\end{aligned}
$$

$$
\begin{aligned}
+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) \\
&(\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})' \\
&+ \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' \\
&+ \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'
\end{aligned}
$$

$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'$$
$$+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')$$

$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)$$
$$(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'$$
$$+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')$$

$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})$$
$$(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'$$
$$+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')$$
$$]$$

$$=E[$$
$$(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})$$
$$(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'$$
$$+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')$$

$$+(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})$$

$$
\begin{aligned}
&\left(\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'\right.\\
&+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'\\
&+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'\\
&+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})'\\
&+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})'\\
&+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})'\\
&\left.+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})'\right)
\end{aligned}
$$

$$
\begin{aligned}
&+(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\\
&\left(\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'\right.\\
&+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'\\
&+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'\\
&+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})'\\
&+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})'\\
&+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})'\\
&\left.+0\right)
\end{aligned}
$$

$$
\begin{aligned}
&+(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})\\
&\left(\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'\right.\\
&+\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'\\
&+\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'\\
&+\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})'\\
&+\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})'\\
&+\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})'\\
&\left.+\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})'\right)
\end{aligned}
$$

$$
\begin{aligned}
&+(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})\\
&\left(\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'\right.\\
&+\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'\\
&+\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'\\
&+\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})'\\
&+\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})'\\
&+\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})'
\end{aligned}
$$

$$+0)$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x}\right)$$
$$(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta}\right)'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x}\right)'$$
$$+0)$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right)$$
$$(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$$
$$+0$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)'$$
$$+0$$
$$+0$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right)')$$
$$]$$

$$= E[$$
$$\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)$$
1) $$(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'$$
2) $$+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$$
3) $$+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'$$
4) $$+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)'$$
5) $$+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta}\right)'$$
6) $$+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x}\right)'$$
7) $$+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right)')$$

$$+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)$$
8) $$(\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'$$
9) $$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$$
10) $$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'$$
11) $$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)'$$

$$12)\quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})'$$

$$13)\quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})'$$

$$14)\quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})')$$

$$+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})$$

$$15)\quad (\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'$$

$$16)\quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$$

$$17)\quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'$$

$$18)\quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})'$$

$$19)\quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})'$$

$$20)\quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})')$$

$$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})$$

$$21)\quad (\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'$$

$$22)\quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$$

$$23)\quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'$$

$$24)\quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})'$$

$$25)\quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})'$$

$$26)\quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})'$$

$$27)\quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})')$$

$$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})$$

$$28)\quad (\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'$$

$$29)\quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$$

$$30)\quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'$$

$$31)\quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})'$$

$$32)\quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})'$$

$$33)\quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})')$$

$$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})$$

$$34)\quad (\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'$$

$$35)\quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$$

$$36)\quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'$$

37) $+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$

38) $+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$

39) $+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})')$

$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})$

40) $((\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$

41) $+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$

42) $+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$

43) $+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')$

]

We next derive how to compute the moments in these terms.

1)

$$E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$=E\left[\left(\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\{\epsilon_{t+1}(\phi_1,1)\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right)\left(\left\{x_t^f(\gamma_3,1)\left\{x_t^f(\gamma_4,1)\{\epsilon_{t+1}(\phi_2,1)\}_{\phi_2=1}^{n_e}\right\}_{\gamma_4=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_xfxfeps_xfxfeps = zeros(nx × nx × ne, nx × nx × ne)
index1 = 0
for gama1 = 1 : nx
    for gama2 = 1 : nx
        for phi1 = 1 : ne
            index1 = index1 + 1
            index2 = 0
            for gama3 = 1 : nx
                for gama4 = 1 : nx
                    for phi2 = 1 : ne
                        index2 = index2 + 1
                        if phi1 == phi2
                            E_xfxfeps_xfxfeps(index1, index2) = E_xf_xf_xf_xf(gama1, gama2, gama3, gama4)
                        end
                    end
                end
            end
        end
    end
end
```

2)

$$E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$$

$$= E\left[\left(\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\{\epsilon_{t+1}(\phi_1,1)\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right)\left(\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_4,1)\right\}_{\gamma_4=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_xfxfeps_xfepsxf = zeros(nx × nx × ne, nx × ne × nx)
index1 = 0
for gama1 = 1 : nx
    for gama2 = 1 : nx
        for phi1 = 1 : ne
            index1 = index1 + 1
            index2 = 0
            for gama3 = 1 : nx
                for phi2 = 1 : ne
                    for gama4 = 1 : nx
                        index2 = index2 + 1
                        if phi1 == phi2
                            E_xfxfeps_xfepsxf(index1, index2)
                            = E_xf_xf_xf_xf(gama1, gama2, gama3, gama4)
                        end
                    end
                end
            end
        end
    end
end
```

3)

$$E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left(\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\{\epsilon_{t+1}(\phi_1,1)\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right)\left(\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_2,1)\{\epsilon_{t+1}(\phi_3,1)\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_xfxfeps_xfeps2 = zeros(nx × nx × ne, nx × ne × ne)
index1 = 0
for gama1 = 1 : nx
    for gama2 = 1 : nx
        for phi1 = 1 : ne
            index1 = index1 + 1
            index2 = 0
            for gama3 = 1 : nx
                for phi2 = 1 : ne
                    for phi3 = 1 : ne
                        index2 = index2 + 1
                        if phi1 == phi2 && phi1 == phi3
                            E_xfxfeps_xfeps2(index1, index2)
                            = E_xf_xf_xf(gama1, gama2, gama3) × m^3 (ϵ_{t+1} (phi1))
                        end
                    end
                end
            end
```

end
end
end

4)
$E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$

$$= E\left[\left(\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_1,1)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right)\left(\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_3,1)\left\{x_t^f(\gamma_4,1)\right\}_{\gamma_4=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are:
$E\_xfxfeps\_epsxfxf = zeros(nx \times nx \times ne, ne \times nx \times nx)$
$index1 = 0$
$for\ gama1 = 1 : nx$
$for\ gama2 = 1 : nx$
$for\ phi1 = 1 : ne$
$index1 = index1 + 1$
$index2 = 0$
$for\ phi2 = 1 : ne$
$for\ gama3 = 1 : nx$
$for\ gama4 = 1 : nx$
$index2 = index2 + 1$
$if\ phi1 == phi2$
$E\_xfxfeps\_epsxfxf(index1, index2)$
$= E\_xf\_xf\_xf\_xf(gama1, gama2, gama3, gama4)$
end
end
end
end
end
end
end

5)
$E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$= E\left[\left(\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_1,1)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right)\left(\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_3,1)\right\}_{\phi_3=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are:
$E\_xfxfeps\_epsxfeps = zeros(nx \times nx \times ne, ne \times nx \times ne)$
$index1 = 0$
$for\ gama1 = 1 : nx$
$for\ gama2 = 1 : nx$
$for\ phi1 = 1 : ne$
$index1 = index1 + 1$
$index2 = 0$
$for\ phi2 = 1 : ne$
$for\ gama3 = 1 : nx$
$for\ phi3 = 1 : ne$

$index2 = index2 + 1$
$if\ phi1 == phi2\ \&\&\ phi1 == phi3$
$E\_xfxfeps\_epsxfeps(index1, index2)$
$= E\_xf\_xf\_xf(gama1, gama2, gama3) \times m^3\left(\epsilon_{t+1}\left(phi1\right)\right)$
end
end
end
end
end
end
end

6)
$E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$

$$= E\left[\left(\left\{x_t^f\left(\gamma_1, 1\right)\left\{x_t^f\left(\gamma_2, 1\right)\left\{\epsilon_{t+1}\left(\phi_1, 1\right)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right)\left(\left\{\epsilon_{t+1}\left(\phi_2, 1\right)\left\{\epsilon_{t+1}\left(\phi_3, 1\right)\left\{x_t^f\left(\gamma_3, 1\right)\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are:
$E\_xfxfeps\_eps2xf = zeros(nx \times nx \times ne, ne \times ne \times nx)$
$index1 = 0$
$for\ gama1 = 1 : nx$
$for\ gama2 = 1 : nx$
$for\ phi1 = 1 : ne$
$index1 = index1 + 1$
$index2 = 0$
$for\ phi2 = 1 : ne$
$for\ phi3 = 1 : ne$
$for\ gama3 = 1 : nx$
$index2 = index2 + 1$
$if\ phi1 == phi2\ \&\&\ phi1 == phi3$
$E\_xfxfeps\_eps2xf(index1, index2)$
$= E\_xf\_xf\_xf(gama1, gama2, gama3) \times m^3\left(\epsilon_{t+1}\left(phi1\right)\right)$
end
end
end
end
end
end
end

7)
$E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$= E[\left(\left\{x_t^f\left(\gamma_1, 1\right)\left\{x_t^f\left(\gamma_2, 1\right)\left\{\epsilon_{t+1}\left(\phi_1, 1\right)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right)$$
$$\times\left(\left\{\epsilon_{t+1}\left(\phi_2, 1\right)\left\{\epsilon_{t+1}\left(\phi_3, 1\right)\left\{\epsilon_{t+1}\left(\phi_4, 1\right)\right\}_{\phi_4=1}^{n_e}\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right)']$$

Thus the quasi Matlab codes are:

$E\_xfxfeps\_eps3 = zeros(nx \times nx \times ne, ne \times ne \times ne)$
$index1 = 0$
$for\ gama1 = 1 : nx$
    $for\ gama2 = 1 : nx$
        $for\ phi1 = 1 : ne$
            $index1 = index1 + 1$
            $index2 = 0$
            $for\ phi2 = 1 : ne$
                $for\ phi3 = 1 : ne$
                    $for\ phi4 = 1 : ne$
                        $index2 = index2 + 1$
                        % second moments of innovations
                        $if\ (phi1 == phi2\ \&\&\ phi3 == phi4\ \&\&\ phi1\tilde{} = phi4)$
                            $E\_xfxfeps\_eps3(index1, index2) = E\_xf\_xf(gama1, gama2)$
                        $elseif\ (phi1 == phi3\ \&\&\ phi2 == phi4\ \&\&\ phi1\tilde{} = phi2)$
                            $E\_xfxfeps\_eps3(index1, index2) = E\_xf\_xf(gama1, gama2)$
                        $elseif\ (phi1 == phi4\ \&\&\ phi2 == phi3\ \&\&\ phi1\tilde{} = phi2)$
                            $E\_xfxfeps\_eps3(index1, index2) = E\_xf\_xf(gama1, gama2)$
                        % fourth moments of innovations
                        $elseif(phi1 == phi2\ \&\&\ phi1 == phi3\ \&\&\ phi1 == phi4)$
                            $E\_xfxfeps\_eps3(index1, index2) = E\_xf\_xf(gama1, gama2) \times m^4\left(\epsilon_{t+1}\left(phi1\right)\right)$
                        end
                    end
                end
            end
        end
    end
end

8)
$$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right] = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]'$$
where we aldready know $E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$ from 2).

9)
$$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$$

$$= E[\left(\left\{x_t^f\left(\gamma_1, 1\right)\left\{\epsilon_{t+1}\left(\phi_1, 1\right)\left\{x_t^f\left(\gamma_2, 1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)$$
$$\times\left(\left\{x_t^f\left(\gamma_3, 1\right)\left\{\epsilon_{t+1}\left(\phi_2, 1\right)\left\{x_t^f\left(\gamma_4, 1\right)\right\}_{\gamma_4=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right)']$$
Thus the quasi Matlab codes are:
$E\_xfepsxf\_xfepsxf = zeros(nx \times ne \times nx, nx \times ne \times nx)$
$index1 = 0$
$for\ gama1 = 1 : nx$
    $for\ phi1 = 1 : ne$
        $for\ gama2 = 1 : nx$

$index1 = index1 + 1$
$index2 = 0$
$for\ gama3 = 1 : nx$
$for\ phi2 = 1 : ne$
$for\ gama4 = 1 : nx$
$index2 = index2 + 1$
$if\ phi1 == phi2$
$E\_xfepsxf\_xfepsxf(index1, index2)$
$= E\_xf\_xf\_xf\_xf(gama1, gama2, gama3, gama4)$
end
end
end
end
end
end
end

10)

$$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E[\left(\left\{x_t^f(\gamma_1,1)\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)$$
$$\times\left(\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{\epsilon_{t+1}(\phi_3,1)\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right)']$$

Thus the quasi Matlab codes are:
$E\_xfepsxf\_xfeps2 = zeros(nx \times ne \times nx, nx \times ne \times ne)$
$index1 = 0$
$for\ gama1 = 1 : nx$
$for\ phi1 = 1 : ne$
$for\ gama2 = 1 : nx$
$index1 = index1 + 1$
$index2 = 0$
$for\ gama3 = 1 : nx$
$for\ phi2 = 1 : ne$
$for\ phi3 = 1 : ne$
$index2 = index2 + 1$
$if\ phi1 == phi2\ \&\&\ phi1 == phi3$
$E\_xfepsxf\_xfeps2(index1, index2)$
$= E\_xf\_xf\_xf(gama1, gama2, gama3) \times m^3(\epsilon_{t+1}(phi1))$
end
end
end
end
end
end
end

11)

$$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$$

$$= E[\left(\left\{x_t^f\left(\gamma_1,1\right)\left\{\epsilon_{t+1}\left(\phi_1,1\right)\left\{x_t^f\left(\gamma_2,1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right) \times \left(\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{x_t^f\left(\gamma_3,1\right)\left\{x_t^f\left(\gamma_4,1\right)\right\}_{\gamma_4=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)']$$

Thus the quasi Matlab codes are:

```
E_xfepsxf_epsxfxf = zeros(nx × ne × nx, ne × nx × nx)
index1 = 0
for gama1 = 1 : nx
    for phi1 = 1 : ne
        for gama2 = 1 : nx
            index1 = index1 + 1
            index2 = 0
            for phi2 = 1 : ne
                for gama3 = 1 : nx
                    for gama4 = 1 : nx
                        index2 = index2 + 1
                        if phi1 == phi2
                            E_xfepsxf_epsxfxf(index1, index2)
                            = E_xf_xf_xf_xf(gama1, gama2, gama3, gama4)
                        end
                    end
                end
            end
        end
    end
end
```

12)

$$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E[\left(\left\{x_t^f\left(\gamma_1,1\right)\left\{\epsilon_{t+1}\left(\phi_1,1\right)\left\{x_t^f\left(\gamma_2,1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right) \times \left(\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{x_t^f\left(\gamma_3,1\right)\left\{\epsilon_{t+1}\left(\phi_3,1\right)\right\}_{\phi_3=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)']$$

Thus the quasi Matlab codes are:

```
E_xfepsxf_epsxfeps = zeros(nx × ne × nx, ne × nx × ne)
index1 = 0
for gama1 = 1 : nx
    for phi1 = 1 : ne
        for gama2 = 1 : nx
            index1 = index1 + 1
            index2 = 0
            for phi2 = 1 : ne
                for gama3 = 1 : nx
```

$for\ phi3 = 1 : ne$
$index2 = index2 + 1$
$if\ phi1 == phi2\ \&\&\ phi1 == phi3$
$E\_xfepsxf\_epsxfeps(index1, index2)$
$= E\_xf\_xf\_xf(gama1, gama2, gama3) \times m^3 (\epsilon_{t+1} (phi1))$
end
end
end
end
end
end
end

13)
$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$

$$= E[\left(\left\{x_t^f (\gamma_1, 1) \left\{\epsilon_{t+1} (\phi_1, 1) \left\{x_t^f (\gamma_2, 1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)$$
$$\times \left(\left\{\epsilon_{t+1} (\phi_2, 1) \left\{\epsilon_{t+1} (\phi_3, 1) \left\{x_t^f (\gamma_3, 1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right)']$$

Thus the quasi Matlab codes are:
$E\_xfepsxf\_eps2xf = zeros(nx \times ne \times nx, ne \times ne \times nx)$
$index1 = 0$
$for\ gama1 = 1 : nx$
$for\ phi1 = 1 : ne$
$for\ gama2 = 1 : nx$
$index1 = index1 + 1$
$index2 = 0$
$for\ phi2 = 1 : ne$
$for\ phi3 = 1 : ne$
$for\ gama3 = 1 : nx$
$index2 = index2 + 1$
$if\ phi1 == phi2\ \&\&\ phi1 == phi3$
$E\_xfepsxf\_eps2xf(index1, index2)$
$= E\_xf\_xf\_xf(gama1, gama2, gama3) \times m^3 (\epsilon_{t+1} (phi1))$
end
end
end
end
end
end
end

14)
$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$= E[\left(\left\{x_t^f (\gamma_1, 1) \left\{\epsilon_{t+1} (\phi_1, 1) \left\{x_t^f (\gamma_2, 1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)$$

$$\times\left(\left\{\epsilon_{t+1}(\phi_2,1)\left\{\epsilon_{t+1}(\phi_3,1)\left\{\epsilon_{t+1}(\phi_4,1)\right\}_{\phi_4=1}^{n_e}\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right)']$$

Thus the quasi Matlab codes are:

```
E_xfepsxf_eps3 = zeros(nx × ne × nx, ne × ne × ne)
index1 = 0
for gama1 = 1 : nx
    for phi1 = 1 : ne
        for gama2 = 1 : nx
            index1 = index1 + 1
            index2 = 0
            for phi2 = 1 : ne
                for phi3 = 1 : ne
                    for phi4 = 1 : ne
                        index2 = index2 + 1
                        % second moments of innovations
                        if (phi1 == phi2 && phi3 == phi4 && phi1~ = phi4)
                            E_xfepsxf_eps3(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi3 && phi2 == phi4 && phi1~ = phi2)
                            E_xfepsxf_eps3(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi4 && phi2 == phi3 && phi1~ = phi2)
                            E_xfepsxf_eps3(index1, index2) = E_xf_xf(gama1, gama2)
                        % fourth moments of innovations
                        if phi1 == phi2 && phi1 == phi3 && phi1 == phi4
                            E_xfepsxf_eps3(index1, index2) = E_xf_xf(gama1, gama2) × m^4 (ε_{t+1} (phi1))
                        end
                    end
                end
            end
        end
    end
end
```

15)

$$E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]=E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]'$$

where we already know $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$ from 3).

16)

$$E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]=E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]'$$

where we already know $E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$ from 10)

17)

$$E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$=E[\left(\left\{x_t^f(\gamma_1,1)\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)$$

$$\times\left(\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_3,1)\left\{\epsilon_{t+1}(\phi_4,1)\right\}_{\phi_4=1}^{n_e}\right\}_{\phi_3=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right)']$$

Thus the quasi Matlab codes are:

```
E_xfeps2_xfeps2 = zeros(nx × ne × ne, nx × ne × ne)
index1 = 0
for gama1 = 1 : nx
    for phi1 = 1 : ne
        for phi2 = 1 : ne
            index1 = index1 + 1
            index2 = 0
            for gama2 = 1 : nx
                for phi3 = 1 : ne
                    for phi4 = 1 : ne
                        index2 = index2 + 1
                        % second moments of innovations
                        if (phi1 == phi2 && phi3 == phi4 && phi1~ = phi4)
                            E_xfeps2_xfeps2(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi3 && phi2 == phi4 && phi1~ = phi2)
                            E_xfeps2_xfeps2(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi4 && phi2 == phi3 && phi1~ = phi2)
                            E_xfeps2_xfeps2(index1, index2) = E_xf_xf(gama1, gama2)
                        % fourth moments of innovations
                        elseif phi1 == phi2 && phi1 == phi3 && phi1 == phi4
                            E_xfeps2_xfeps2(index1, index2) = E_xf_xf(gama1, gama2) × m^4 (ε_{t+1} (phi1))
                        end
                    end
                end
            end
        end
    end
end
```

18)

$$E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]$$

$$=E[\left(\left\{x_t^f(\gamma_1,1)\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)$$
$$\times\left(\left\{\epsilon_{t+1}(\phi_3,1)\left\{x_t^f(\gamma_2,1)\left\{x_t^f(\gamma_3,1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_3=1}^{n_e}\right)']$$

Thus the quasi Matlab codes are:

```
E_xfeps2_epsxfxf = zeros(nx × ne × ne, ne × nx × nx)
index1 = 0
for gama1 = 1 : nx
    for phi1 = 1 : ne
        for phi2 = 1 : ne
            index1 = index1 + 1
            index2 = 0
            for phi3 = 1 : ne
```

$for\ gama2 = 1 : nx$
$for\ gama3 = 1 : nx$
$index2 = index2 + 1$
$if\ (phi1 == phi2\ \&\&\ phi1 == phi3$
$E\_xfeps2\_epsxfxf(index1, index2)$
$= E\_xf\_xf\_xf(gama1, gama2, gama3) \times m^3 (\epsilon_{t+1} (phi1))$
end
end
end
end
end
end
end

19)
$E \left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \right)' \right]$

$$= E[\left(\left\{ x_t^f (\gamma_1, 1) \left\{ \epsilon_{t+1} (\phi_1, 1) \left\{ \epsilon_{t+1} (\phi_2, 1) \right\}_{\phi_2=1}^{n_e} \right\}_{\phi_1=1}^{n_e} \right\}_{\gamma_1=1}^{n_x} \right) \times \left(\left\{ \epsilon_{t+1} (\phi_3, 1) \left\{ x_t^f (\gamma_2, 1) \left\{ \epsilon_{t+1} (\phi_4, 1) \right\}_{\phi_4=1}^{n_e} \right\}_{\gamma_2=1}^{n_x} \right\}_{\phi_3=1}^{n_e} \right)']$$

Thus the quasi Matlab codes are:
$E\_xfeps2\_epsxfeps = zeros(nx \times ne \times ne, ne \times nx \times ne)$
$index1 = 0$
$for\ gama1 = 1 : nx$
$for\ phi1 = 1 : ne$
$for\ phi2 = 1 : ne$
$index1 = index1 + 1$
$index2 = 0$
$for\ phi3 = 1 : ne$
$for\ gama2 = 1 : nx$
$for\ phi4 = 1 : ne$
$index2 = index2 + 1$
% second moments of innovations
$if\ (phi1 == phi2\ \&\&\ phi3 == phi4\ \&\&\ phi1\tilde{} = phi4)$
$E\_xfeps2\_epsxfeps(index1, index2) = E\_xf\_xf(gama1, gama2)$
$elseif\ (phi1 == phi3\ \&\&\ phi2 == phi4\ \&\&\ phi1\tilde{} = phi2)$
$E\_xfeps2\_epsxfeps(index1, index2) = E\_xf\_xf(gama1, gama2)$
$elseif\ (phi1 == phi4\ \&\&\ phi2 == phi3\ \&\&\ phi1\tilde{} = phi2)$
$E\_xfeps2\_epsxfeps(index1, index2) = E\_xf\_xf(gama1, gama2)$
% fourth moments of innovations
$elseif\ phi1 == phi2\ \&\&\ phi1 == phi3\ \&\&\ phi1 == phi4$
$E\_xfeps2\_epsxfeps(index1, index2) = E\_xf\_xf(gama1, gama2) \times m^4 (\epsilon_{t+1} (phi1))$
end
end
end
end
end

end
end

20)
$$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$$

$$= E[\left(\left\{x_t^f(\gamma_1, 1)\left\{\epsilon_{t+1}(\phi_1, 1)\left\{\epsilon_{t+1}(\phi_2, 1)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)$$
$$\times\left(\left\{\epsilon_{t+1}(\phi_3, 1)\left\{\epsilon_{t+1}(\phi_4, 1)\left\{x_t^f(\gamma_2, 1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_4=1}^{n_e}\right\}_{\phi_3=1}^{n_e}\right)']$$

Thus the quasi Matlab codes are:

```
E_xfeps2_eps2xf = zeros(nx × ne × ne, ne × ne × nx)
index1 = 0
for gama1 = 1 : nx
    for phi1 = 1 : ne
        for phi2 = 1 : ne
            index1 = index1 + 1
            index2 = 0
            for phi3 = 1 : ne
                for phi4 = 1 : ne
                    for gama2 = 1 : nx
                        index2 = index2 + 1
                        % second moments of innovations
                        if (phi1 == phi2 && phi3 == phi4 && phi1˜ = phi4)
                            E_xfeps2_eps2xf(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi3 && phi2 == phi4 && phi1˜ = phi2)
                            E_xfeps2_eps2xf(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi4 && phi2 == phi3 && phi1˜ = phi2)
                            E_xfeps2_eps2xf(index1, index2) = E_xf_xf(gama1, gama2)
                        % fourth moments of innovations
                        elseif phi1 == phi2 && phi1 == phi3 && phi1 == phi4
                            E_xfeps2_eps2xf(index1, index2) = E_xf_xf(gama1, gama2) × m^4 (ε_{t+1} (phi1))
                        end
                    end
                end
            end
        end
    end
end
```

21)
$$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right] = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]'$$

where we already know $E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$ from 4).

22)
$$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right] = E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]'$$

where we already know $E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$ from 11)

23)
$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right] = E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]'$
where we already know $E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$ from 18)

24)
$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right] =$

$$= E[\left(\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)$$
$$\times\left(\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_3,1)\left\{x_t^f(\gamma_4,1)\right\}_{\gamma_4=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)']$$

Thus the quasi Matlab codes are:

```
E_epsxfxf_epsxfxf = zeros(ne × nx × nx, ne × nx × nx)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        for gama2 = 1 : nx
            index1 = index1 + 1
            index2 = 0
            for phi2 = 1 : ne
                for gama3 = 1 : nx
                    for gama4 = 1 : nx
                        index2 = index2 + 1
                        if (phi1 == phi2
                            E_epsxfxf_epsxfxf(index1, index2)
                            = E_xf_xf_xf_xf(gama1, gama2, gama3, gama4)
                        end
                    end
                end
            end
        end
    end
end
```

25)
$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right] =$

$$= E[\left(\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)$$

$$\times \left(\left\{ \epsilon_{t+1} (\phi_2, 1) \left\{ x_t^f (\gamma_3, 1) \{ \epsilon_{t+1} (\phi_3, 1) \}_{\phi_3=1}^{n_e} \right\}_{\gamma_3=1}^{n_x} \right\}_{\phi_2=1}^{n_e} \right)']$$

Thus the quasi Matlab codes are:

$E\_epsxfxf\_epsxfeps = zeros(ne \times nx \times nx, ne \times nx \times ne)$
$index1 = 0$
$for\ phi1 = 1 : ne$
$\quad for\ gama1 = 1 : nx$
$\quad\quad for\ gama2 = 1 : nx$
$\quad\quad\quad index1 = index1 + 1$
$\quad\quad\quad index2 = 0$
$\quad\quad\quad for\ phi2 = 1 : ne$
$\quad\quad\quad\quad for\ gama3 = 1 : nx$
$\quad\quad\quad\quad\quad for\ phi3 = 1 : ne$
$\quad\quad\quad\quad\quad\quad index2 = index2 + 1$
$\quad\quad\quad\quad\quad\quad if\ (phi1 == phi2\ \&\&\ phi1 == phi3$
$\quad\quad\quad\quad\quad\quad\quad E\_epsxfxf\_epsxfeps(index1, index2)$
$\quad\quad\quad\quad\quad\quad\quad = E\_xf\_xf\_xf(gama1, gama2, gama3) \times m^3 (\epsilon_{t+1} (phi1))$
$\quad\quad\quad\quad\quad\quad end$
$\quad\quad\quad\quad\quad end$
$\quad\quad\quad\quad end$
$\quad\quad\quad end$
$\quad\quad end$
$\quad end$
end

26)

$$E \left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f \right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \right)' \right]$$

$$= E[\left(\left\{ \epsilon_{t+1} (\phi_1, 1) \left\{ x_t^f (\gamma_1, 1) \left\{ x_t^f (\gamma_2, 1) \right\}_{\gamma_2=1}^{n_x} \right\}_{\gamma_1=1}^{n_x} \right\}_{\phi_1=1}^{n_e} \right)$$
$$\times \left(\left\{ \epsilon_{t+1} (\phi_2, 1) \left\{ \epsilon_{t+1} (\phi_3, 1) \left\{ x_t^f (\gamma_3, 1) \right\}_{\gamma_3=1}^{n_x} \right\}_{\phi_3=1}^{n_e} \right\}_{\phi_2=1}^{n_e} \right)']$$

Thus the quasi Matlab codes are:

$E\_epsxfxf\_eps2xf = zeros(ne \times nx \times nx, ne \times ne \times nx)$
$index1 = 0$
$for\ phi1 = 1 : ne$
$\quad for\ gama1 = 1 : nx$
$\quad\quad for\ gama2 = 1 : nx$
$\quad\quad\quad index1 = index1 + 1$
$\quad\quad\quad index2 = 0$
$\quad\quad\quad for\ phi2 = 1 : ne$
$\quad\quad\quad\quad for\ phi3 = 1 : ne$
$\quad\quad\quad\quad\quad for\ gama3 = 1 : nx$
$\quad\quad\quad\quad\quad\quad index2 = index2 + 1$
$\quad\quad\quad\quad\quad\quad if\ (phi1 == phi2\ \&\&\ phi1 == phi3$
$\quad\quad\quad\quad\quad\quad\quad E\_epsxfxf\_eps2xf(index1, index2)$
$\quad\quad\quad\quad\quad\quad\quad = E\_xf\_xf\_xf(gama1, gama2, gama3) \times m^3 (\epsilon_{t+1} (phi1))$
$\quad\quad\quad\quad\quad\quad end$

end
end
end
end
end
end

27)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$=E[\left(\left\{\epsilon_{t+1}\left(\phi_1,1\right)\left\{x_t^f\left(\gamma_1,1\right)\left\{x_t^f\left(\gamma_2,1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)$$
$$\times\left(\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{\epsilon_{t+1}\left(\phi_3,1\right)\left\{\epsilon_{t+1}\left(\phi_4,1\right)\right\}_{\phi_4=1}^{n_e}\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right)']$$

Thus the quasi Matlab codes are:

```
E_epsxfxf_eps3 = zeros(ne × nx × nx, ne × ne × ne)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        for gama2 = 1 : nx
            index1 = index1 + 1
            index2 = 0
            for phi2 = 1 : ne
                for phi3 = 1 : ne
                    for phi4 = 1 : ne
                        index2 = index2 + 1
                        % second moments of innovations
                        if (phi1 == phi2 && phi3 == phi4 && phi1˜ = phi4)
                            E_epsxfxf_eps3(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi3 && phi2 == phi4 && phi1˜ = phi2)
                            E_epsxfxf_eps3(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi4 && phi2 == phi3 && phi1˜ = phi2)
                            E_epsxfxf_eps3(index1, index2) = E_xf_xf(gama1, gama2)
                        % fourth moments of innovations
                        elseif phi1 == phi2 && phi1 == phi3 && phi1 == phi4
                            E_epsxfxf_eps3(index1, index2) = E_xf_xf(gama1, gama2) × m^4 (ε_{t+1} (phi1))
                        end
                    end
                end
            end
        end
    end
end
```

28)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]=E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]'$
where we already know $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$ from 5).

29)

$$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]=E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]'$$

where we already know $E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$ from 2)

30)

$$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]=E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]'$$

where we already know $E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$ from 19).

31)

$$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]=E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]'$$

where we already know $E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$ from 25).

32)

$$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$=E[\left(\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)$$
$$\times\left(\left\{\epsilon_{t+1}(\phi_3,1)\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_4,1)\right\}_{\phi_4=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_3=1}^{n_e}\right)']$$

Thus the quasi Matlab codes are:

```
E_epsxfeps_epsxfeps = zeros(ne × nx × ne, ne × nx × ne)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        for phi2 = 1 : ne
            index1 = index1 + 1
            index2 = 0
            for phi3 = 1 : ne
                for gama2 = 1 : nx
                    for phi4 = 1 : ne
                        index2 = index2 + 1
                        % second moments of innovations
                        if (phi1 == phi2 && phi3 == phi4 && phi1~ = phi4)
                            E_epsxfeps_epsxfeps(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi3 && phi2 == phi4 && phi1~ = phi2)
                            E_epsxfeps_epsxfeps(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi4 && phi2 == phi3 && phi1~ = phi2)
                            E_epsxfeps_epsxfeps(index1, index2) = E_xf_xf(gama1, gama2)
                        % fourth moments of innovations
                        elseif phi1 == phi2 && phi1 == phi3 && phi1 == phi4
                            E_epsxfeps_epsxfeps(index1, index2) = E_xf_xf(gama1, gama2)×m^4(ε_{t+1}(phi1))
                        end
```

end
end
end
end
end
end

33)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$

$$=E[\left(\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)$$
$$\times\left(\left(\left\{\epsilon_{t+1}(\phi_3,1)\left\{\epsilon_{t+1}(\phi_4,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_4=1}^{n_e}\right\}_{\phi_3=1}^{n_e}\right)\right)']$$

Thus the quasi Matlab codes are:

$E\_epsxfeps\_eps2xf = zeros(ne\times nx\times ne, ne\times ne\times nx)$
$index1 = 0$
$for\ phi1 = 1 : ne$
$\quad for\ gama1 = 1 : nx$
$\quad\quad for\ phi2 = 1 : ne$
$\quad\quad\quad index1 = index1 + 1$
$\quad\quad\quad index2 = 0$
$\quad\quad\quad for\ phi3 = 1 : ne$
$\quad\quad\quad\quad for\ phi4 = 1 : ne$
$\quad\quad\quad\quad\quad for\ gama2 = 1 : nx$
$\quad\quad\quad\quad\quad\quad index2 = index2 + 1$
$\quad\quad\quad\quad\quad\quad$ % second moments of innovations
$\quad\quad\quad\quad\quad\quad if\ (phi1 == phi2\ \&\&\ phi3 == phi4\ \&\&\ phi1\tilde{} = phi4)$
$\quad\quad\quad\quad\quad\quad\quad E\_epsxfeps\_eps2xf(index1, index2) = E\_xf\_xf(gama1, gama2)$
$\quad\quad\quad\quad\quad\quad elseif\ (phi1 == phi3\ \&\&\ phi2 == phi4\ \&\&\ phi1\tilde{} = phi2)$
$\quad\quad\quad\quad\quad\quad\quad E\_epsxfeps\_eps2xf(index1, index2) = E\_xf\_xf(gama1, gama2)$
$\quad\quad\quad\quad\quad\quad elseif\ (phi1 == phi4\ \&\&\ phi2 == phi3\ \&\&\ phi1\tilde{} = phi2)$
$\quad\quad\quad\quad\quad\quad\quad E\_epsxfeps\_eps2xf(index1, index2) = E\_xf\_xf(gama1, gama2)$
$\quad\quad\quad\quad\quad\quad$ % fourth moments of innovations
$\quad\quad\quad\quad\quad\quad elseif\ phi1 == phi2\ \&\&\ phi1 == phi3\ \&\&\ phi1 == phi4$
$\quad\quad\quad\quad\quad\quad\quad E\_epsxfeps\_eps2xf(index1, index2) = E\_xf\_xf(gama1, gama2)\times m^4(\epsilon_{t+1}(phi1))$
$\quad\quad\quad\quad\quad\quad end$
$\quad\quad\quad\quad\quad end$
$\quad\quad\quad\quad end$
$\quad\quad\quad end$
$\quad\quad end$
$\quad end$
end

34)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]=E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]'$

where we already know $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$ from 6).

35)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]=E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]'$
where we know $E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$ from 13).

36)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]=E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]'$
where we already know $E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$ from 20).

37)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]=E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]'$
where we already know $E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$ from 26).

38)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]=E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]'$
where we already know $E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$ from 33).

39)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$

$$=E[\left(\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right)$$
$$\times\left(\left\{\epsilon_{t+1}(\phi_3,1)\left\{\epsilon_{t+1}(\phi_4,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_4=1}^{n_e}\right\}_{\phi_3=1}^{n_e}\right)']$$

Thus the quasi Matlab codes are:

```
E_eps2xf_eps2xf = zeros(ne × ne × nx, ne × ne × nx)
index1 = 0
for phi1 = 1 : ne
    for phi2 = 1 : ne
        for gama1 = 1 : nx
            index1 = index1 + 1
            index2 = 0
            for phi3 = 1 : ne
                for phi4 = 1 : ne
                    for gama2 = 1 : nx
                        index2 = index2 + 1
                        % second moments of innovations
                        if (phi1 == phi2 && phi3 == phi4 && phi1~ = phi4)
                            E_eps2xf_eps2xf(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi3 && phi2 == phi4 && phi1~ = phi2)
                            E_eps2xf_eps2xf(index1, index2) = E_xf_xf(gama1, gama2)
```

$elseif\ (phi1 == phi4\ \&\&\ phi2 == phi3\ \&\&\ phi1\tilde{}= phi2)$

$E\_eps2xf\_eps2xf(index1, index2) = E\_xf\_xf(gama1, gama2)$

% fourth moments of innovations

$elseif\ phi1 == phi2\ \&\&\ phi1 == phi3\ \&\&\ phi1 == phi4$

$E\_eps2xf\_eps2xf(index1, index2) = E\_xf\_xf(gama1, gama2) \times m^4\left(\epsilon_{t+1}\left(phi1\right)\right)$

end

end

end

end

end

end

end

40)

$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right] = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]'$

where we already know $E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$ from 7).

41)

$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right] = E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]'$

where we already know $E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$ from 14).

42)

$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right] = E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]'$

where we already know $E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$ from 27).

43)

$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$= E[\left(\left\{\epsilon_{t+1}\left(\phi_1, 1\right)\left\{\epsilon_{t+1}\left(\phi_2, 1\right)\left\{\epsilon_{t+1}\left(\phi_3, 1\right)\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right) \times \left(\left\{\epsilon_{t+1}\left(\phi_4, 1\right)\left\{\epsilon_{t+1}\left(\phi_5, 1\right)\left\{\epsilon_{t+1}\left(\phi_6, 1\right)\right\}_{\phi_6=1}^{n_e}\right\}_{\phi_5=1}^{n_e}\right\}_{\phi_4=1}^{n_e}\right)']$$

The codes are given in the matlab file. (too big for displaying)

4.4 Method 3: Simple formulas for first and second moments

This section shows how to compute mean values up to third order in a very direct manner. As in the case of the second-order approximation, the advantage of Method 3 is that we do not recompute terms which are already known at a lower approximation order. As a result, the matrices which must be inverted are here smaller than in Method 1 and 2.

4.4.1 First moments

This section derives the unconditional mean value of $\mathbf{y}_t$ and $\mathbf{x}_t$. Recall

$$\mathbf{x}_t = \mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}$$

Thus, we only need to find $E\left[\mathbf{x}_t^{rd}\right]$. Here

$E\left[\mathbf{x}_{t+1}^{rd}\right] = \mathbf{h_x} E\left[\mathbf{x}_t^{rd}\right] + 2\tilde{\mathbf{H}}_{\mathbf{xx}} E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\right] + \tilde{\mathbf{H}}_{\mathbf{xxx}} E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] + \frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2 E\left[\mathbf{x}_t^f\right] + \frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$

$\Updownarrow$

$\left(\mathbf{I}_{n_x} - \mathbf{h_x}\right) E\left[\mathbf{x}_t^{rd}\right] = 2\tilde{\mathbf{H}}_{\mathbf{xx}} E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\right] + \tilde{\mathbf{H}}_{\mathbf{xxx}} E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] + \frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$

because $\mathbf{x}_{t+1}^{rd}$ is stationary and $E\left[\mathbf{x}_t^f\right] = \mathbf{0}$

$\Updownarrow$

$$E\left[\mathbf{x}_t^{rd}\right] = \left(\mathbf{I}_{n_x} - \mathbf{h_x}\right)^{-1}\left(2\tilde{\mathbf{H}}_{\mathbf{xx}} E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\right] + \tilde{\mathbf{H}}_{\mathbf{xxx}} E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] + \frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\right)$$

To compute $\tilde{\mathbf{H}}_{\mathbf{xxx}} E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right]$ recall that

$$\begin{aligned}\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f &= \left(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\\ &+ \left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\\ &+ \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\\ &+ \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\end{aligned}$$

Hence

$E\left[\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right] = \left(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \mathbf{h_x}\right) E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] + \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta}\right) E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\right]$

$\Updownarrow$

$\left(\mathbf{I}_{n_x^3} - \left(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \mathbf{h_x}\right)\right) E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right] = \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta}\right) E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\right]$

because $\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f$ is stationary

$\Updownarrow$

$$E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right] = \left(\mathbf{I}_{n_x^3} - \left(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \mathbf{h_x}\right)\right)^{-1}\left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta}\right) E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\right]$$

Note that this term is zero if all third moments of $\boldsymbol{\epsilon}_{t+1}$ are zero.

To compute $E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\right]$ recall that

$$\begin{aligned}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s\right) &= \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \left(\mathbf{h_x} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_t^f\\ &+ \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s\right) + \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\end{aligned}$$

So

$E\left[\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s\right)\right] = \left(\mathbf{h_x} \otimes \mathbf{h_x}\right) E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\right] + \left(\mathbf{h_x} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right) E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right]$

$\Updownarrow$

$\left(\mathbf{I}_{n_x^2} - \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\right) E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\right] = \left(\mathbf{h_x} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right) E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right]$

because $\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)$ is stationary
$\Updownarrow$

$$E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right] = \left(\mathbf{I}_{n_x^2} - (\mathbf{h_x} \otimes \mathbf{h_x})\right)^{-1}\left(\mathbf{h_x} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right) E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right]$$

Note that this term is zero if $E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] = \mathbf{0}$, which is the case if all third moments of $\boldsymbol{\epsilon}_{t+1}$ are zero.

For the control variables, we have

$$E\left[\mathbf{y}_t^{rd}\right] = \mathbf{g_x}\left(E\left[\mathbf{x}_t^f\right] + E\left[\mathbf{x}_t^s\right] + E\left[\mathbf{x}_t^{rd}\right]\right) + \tilde{\mathbf{G}}_{\mathbf{xx}}\left(E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] + 2E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\right]\right) + \tilde{\mathbf{G}}_{\mathbf{xxx}} E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right]$$
$$+\tfrac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2 E\left[\mathbf{x}_t^f\right] + \tfrac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$

$$= \mathbf{g_x}\left(E\left[\mathbf{x}_t^s\right] + E\left[\mathbf{x}_t^{rd}\right]\right) + \tilde{\mathbf{G}}_{\mathbf{xx}}\left(E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] + 2E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\right]\right)$$
$$+\tilde{\mathbf{G}}_{\mathbf{xxx}} E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] + \tfrac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \tfrac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$

Now recall, if all third moments of $\boldsymbol{\epsilon}_{t+1}$ are zero, then $\mathbf{g}_{\sigma\sigma\sigma} = \mathbf{0}$ and $\mathbf{h}_{\sigma\sigma\sigma} = \mathbf{0}$. Hence, we have the following

Corollary 1 *The mean value in a third order approximation is identical to the mean value in a second order approximation if all third moments of $\boldsymbol{\epsilon}_{t+1}$ are zero.*

4.4.2 Second moments

We start by noticing that

$$Var\left(\mathbf{z}_t\right) = E\left[\left(\mathbf{z}_t - E\left[\mathbf{z}_t\right]\right)\left(\mathbf{z}_t - E\left[\mathbf{z}_t\right]\right)'\right]$$

$$= E\left[\left(\mathbf{z}_t - E\left[\mathbf{z}_t\right]\right)\left(\mathbf{z}_t' - E\left[\mathbf{z}_t'\right]\right)\right]$$

$$= E\left[\mathbf{z}_t\mathbf{z}_t' - \mathbf{z}_t E\left[\mathbf{z}_t'\right] - E\left[\mathbf{z}_t\right]\mathbf{z}_t' + E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\right]$$

$$= E\left[\mathbf{z}_t\mathbf{z}_t'\right] - E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]$$

and

$$E\left[\mathbf{z}_t\mathbf{z}_t'\right] = E\left[\begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \\ \mathbf{x}_t^{rd} \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{bmatrix}\begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^s\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^{rd}\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \end{bmatrix}\right]$$

$$= E\begin{bmatrix} \mathbf{x}_t^f\left(\mathbf{x}_t^f\right)' & \mathbf{x}_t^f\left(\mathbf{x}_t^s\right)' & \mathbf{x}_t^f\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' & \mathbf{x}_t^f\left(\mathbf{x}_t^{rd}\right)' \\ \mathbf{x}_t^s\left(\mathbf{x}_t^f\right)' & \mathbf{x}_t^s\left(\mathbf{x}_t^s\right)' & \mathbf{x}_t^s\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' & \mathbf{x}_t^s\left(\mathbf{x}_t^{rd}\right)' \\ \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^s\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^{rd}\right)' \\ \mathbf{x}_t^{rd}\left(\mathbf{x}_t^f\right)' & \mathbf{x}_t^{rd}\left(\mathbf{x}_t^s\right)' & \mathbf{x}_t^{rd}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' & \mathbf{x}_t^{rd}\left(\mathbf{x}_t^{rd}\right)' \\ \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\left(\mathbf{x}_t^s\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\left(\mathbf{x}_t^{rd}\right)' \\ \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^s\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^{rd}\right)' \end{bmatrix}$$

$$
\begin{array}{cc}
\mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)' & \mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)' \\
\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)' & \mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)' \\
\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)' \\
\mathbf{x}_t^{rd}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)' & \mathbf{x}_t^{rd}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)' \\
\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)' \\
\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'
\end{array}\right]
$$

Hence, we need to find the following terms:

- $\mathbf{x}_t^f\left(\mathbf{x}_t^{rd}\right)'$, $\mathbf{x}_t^s\left(\mathbf{x}_t^{rd}\right)'$, $\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^{rd}\right)'$, $\mathbf{x}_t^{rd}\left(\mathbf{x}_t^{rd}\right)'$, $\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^{rd}\right)'$, $\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^{rd}\right)'$
- $\mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)'$, $\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)'$, $\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)'$, $\mathbf{x}_t^{rd}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)'$, $\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)'$, $\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)'$
- $\mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'$, $\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'$, $\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'$, $\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)$

All these terms are easy to compute using the procedure outlined above and previous results.

4.5 The auto-correlations

This section derives the auto-correlations for the states and the control variables.

4.5.1 The innovations

We first show that $Cov\left(\boldsymbol{\xi}_{t+1},\boldsymbol{\xi}_{t+1+s}\right)\neq\mathbf{0}$ for $s=1,2,3,...$ To see this recall that

$$
E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1+s}'\right]=E\left[\left[\begin{array}{c}
\boldsymbol{\epsilon}_{t+1} \\
\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right) \\
\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f \\
\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1} \\
\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s \\
\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f \\
\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1} \\
\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f \\
\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} \\
\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1} \\
\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f \\
\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)-E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right]
\end{array}\right]\right.
$$

$$
\begin{array}{l}
\times\left[\begin{array}{lll}\boldsymbol{\epsilon}_{t+1+s}' & \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}-vec\left(\mathbf{I}_{n_e}\right)\right)' & \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\right)'\end{array}\right. \\
\left(\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \quad \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^s\right)' \quad \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\otimes\mathbf{x}_{t+s}^f\right)' \\
\left(\mathbf{x}_{t+s}^f\otimes\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \quad \left(\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\right)' \quad \left(\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \quad \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \\
\left.\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\right)' \quad \left(\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)-E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right]\right)'\ \right]
\end{array}
$$

We now inspect each of the rows in turn. Here, we need the following result that
$\mathbf{x}_{t+1}^f=\mathbf{h_x}\mathbf{x}_t^f+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$

$$\begin{aligned}\mathbf{x}^f_{t+2} &= \mathbf{h_x}\mathbf{x}^f_{t+1} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2}\\ &= \mathbf{h_x}\left(\mathbf{h_x}\mathbf{x}^f_t + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2}\\ &= \mathbf{h}^2_{\mathbf{x}}\mathbf{x}^f_t + \mathbf{h_x}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2}\end{aligned}$$

...

$\mathbf{x}^f_{t+s} = \mathbf{h}^s_{\mathbf{x}}\mathbf{x}^f_t + \sum_{i=1}^s \mathbf{h}^{s-i}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+i}$

1) Row with $\boldsymbol{\epsilon}_{t+1}$

Consider the sub-matrix

$$\begin{aligned}E\{\boldsymbol{\epsilon}_{t+1}\Big[\ &\boldsymbol{\epsilon}'_{t+1+s} \quad (\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s} - vec\,(\mathbf{I}_{n_e}))' \quad \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}^f_{t+s}\right)'\\ &\left(\mathbf{x}^f_{t+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \quad (\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}^s_{t+s})' \quad \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}^f_{t+s}\otimes\mathbf{x}^f_{t+s}\right)'\\ &\left(\mathbf{x}^f_{t+s}\otimes\mathbf{x}^f_{t+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \quad \left(\mathbf{x}^f_{t+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}^f_{t+s}\right)' \quad \left(\mathbf{x}^f_{t+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \quad \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}^f_{t+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\\ &\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}^f_{t+s}\right)' \quad ((\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}) - E\left[(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\right])' \ \Big]\}\end{aligned}$$

$$\begin{aligned}= E\{\boldsymbol{\epsilon}_{t+1}\big[\ &\mathbf{0} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0'\\ &0 \quad 0 \quad \left(\mathbf{x}^f_{t+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \quad \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}^f_{t+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\\ &\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}^f_{t+s}\right)' \quad 0 \ \Big]\}\end{aligned}$$

Hence, we only need to study the term of the form

$E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}^f_{t+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\right]$

$$= E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{h}^s_{\mathbf{x}}\mathbf{x}^f_t + \sum_{i=1}^s \mathbf{h}^{s-i}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+i}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\right]$$

$$= E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{h}^s_{\mathbf{x}}\mathbf{x}^f_t\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s} + \sum_{i=1}^s \mathbf{h}^{s-i}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+i}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\right]$$

$$\begin{aligned}&= E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{h}^s_{\mathbf{x}}\mathbf{x}^f_t\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\right]\\ &+E\left[\boldsymbol{\epsilon}_{t+1}\left(\sum_{i=1}^s \mathbf{h}^{s-i}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+i}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\right]\end{aligned}$$

$$= 0 + E\left[\boldsymbol{\epsilon}_{t+1}\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\right]$$

$$= E\left[(\boldsymbol{\epsilon}_{t+1}\otimes 1)\left(\boldsymbol{\epsilon}'_{t+1}\boldsymbol{\eta}'\sigma\otimes(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s})'\right)\right]$$

$$= E\left[\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\boldsymbol{\eta}'\sigma\otimes(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s})'\right]$$

$$= \mathbf{I}_{n_e}\boldsymbol{\eta}'\sigma\otimes vec\,(\mathbf{I}_{n_e})'$$

2)

To be completed

4.5.2 The covariances

Recall that we have

$$\mathbf{z}_t = \begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \\ \mathbf{x}_t^{rd} \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{bmatrix}$$

$$\mathbf{z}_{t+1} = \mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}$$

$$\mathbf{y}_t^{rd} = \mathbf{D}\mathbf{z}_t + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$

To find the one period auto-covariances, i.e. $Cov\left(\mathbf{z}_{t+1}, \mathbf{z}_t\right)$, we have

$$\begin{aligned} &Cov\left(\mathbf{z}_{t+1}, \mathbf{z}_t\right) \\ &\quad = Cov\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}, \mathbf{z}_t\right) \\ &\quad = \mathbf{A}Cov\left(\mathbf{z}_t, \mathbf{z}_t\right) + \mathbf{B}Cov\left(\boldsymbol{\xi}_{t+1}, \mathbf{z}_t\right) \end{aligned}$$

And for two periods

$$\begin{aligned} &Cov\left(\mathbf{z}_{t+2}, \mathbf{z}_t\right) = Cov\left(\mathbf{c} + \mathbf{A}\mathbf{z}_{t+1} + \mathbf{B}\boldsymbol{\xi}_{t+2}, \mathbf{z}_t\right) \\ &\quad = Cov\left(\mathbf{c} + \mathbf{A}\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}\right) + \mathbf{B}\boldsymbol{\xi}_{t+2}, \mathbf{z}_t\right) \\ &\quad = Cov\left(\mathbf{c} + \mathbf{A}\mathbf{c} + \mathbf{A}^2\mathbf{z}_t + \mathbf{A}\mathbf{B}\boldsymbol{\xi}_{t+1} + \mathbf{B}\boldsymbol{\xi}_{t+2}, \mathbf{z}_t\right) \\ &\quad = Cov\left(\mathbf{A}^2\mathbf{z}_t, \mathbf{z}_t\right) + Cov\left(\mathbf{A}\mathbf{B}\boldsymbol{\xi}_{t+1}, \mathbf{z}_t\right) + Cov\left(\mathbf{B}\boldsymbol{\xi}_{t+2}, \mathbf{z}_t\right) \\ &\quad = \mathbf{A}^2 Cov\left(\mathbf{z}_t, \mathbf{z}_t\right) + \mathbf{A}\mathbf{B}Cov\left(\boldsymbol{\xi}_{t+1}, \mathbf{z}_t\right) + \mathbf{B}Cov\left(\boldsymbol{\xi}_{t+2}, \mathbf{z}_t\right) \end{aligned}$$

or

$$Cov\left(\mathbf{z}_{t+2}, \mathbf{z}_t\right) = \mathbf{A}Cov\left(\mathbf{z}_{t+1}, \mathbf{z}_t\right) + \mathbf{B}Cov\left(\boldsymbol{\xi}_{t+2}, \mathbf{z}_t\right)$$

And for three periods

$$\begin{aligned} &Cov\left(\mathbf{z}_{t+3}, \mathbf{z}_t\right) = Cov\left(\mathbf{c} + \mathbf{A}\mathbf{z}_{t+2} + \mathbf{B}\boldsymbol{\xi}_{t+3}, \mathbf{z}_t\right) \\ &\quad = Cov\left(\mathbf{c} + \mathbf{A}\left(\mathbf{c} + \mathbf{A}\mathbf{c} + \mathbf{A}^2\mathbf{z}_t + \mathbf{A}\mathbf{B}\boldsymbol{\xi}_{t+1} + \mathbf{B}\boldsymbol{\xi}_{t+2}\right) + \mathbf{B}\boldsymbol{\xi}_{t+3}, \mathbf{z}_t\right) \\ &\quad = Cov\left(\mathbf{c} + \mathbf{A}\mathbf{c} + \mathbf{A}^2\mathbf{c} + \mathbf{A}^3\mathbf{z}_t + \mathbf{A}^2\mathbf{B}\boldsymbol{\xi}_{t+1} + \mathbf{A}\mathbf{B}\boldsymbol{\xi}_{t+2} + \mathbf{B}\boldsymbol{\xi}_{t+3}, \mathbf{z}_t\right) \\ &\quad = Cov\left(\mathbf{A}^3\mathbf{z}_t, \mathbf{z}_t\right) + Cov\left(\mathbf{A}^2\mathbf{B}\boldsymbol{\xi}_{t+1}, \mathbf{z}_t\right) + Cov\left(\mathbf{A}\mathbf{B}\boldsymbol{\xi}_{t+2}, \mathbf{z}_t\right) + Cov\left(\mathbf{B}\boldsymbol{\xi}_{t+3}, \mathbf{z}_t\right) \\ &\quad = \mathbf{A}^3 Var\left(\mathbf{z}_t\right) + \mathbf{A}^2\mathbf{B}Cov\left(\boldsymbol{\xi}_{t+1}, \mathbf{z}_t\right) + \mathbf{A}\mathbf{B}Cov\left(\boldsymbol{\xi}_{t+2}, \mathbf{z}_t\right) + \mathbf{B}Cov\left(\boldsymbol{\xi}_{t+3}, \mathbf{z}_t\right) \\ &\quad = \mathbf{A}^3 Var\left(\mathbf{z}_t\right) + \textstyle\sum_{i=1}^{3} \mathbf{A}^{3-i}\mathbf{B}Cov\left(\boldsymbol{\xi}_{t+i}, \mathbf{z}_t\right) \end{aligned}$$

or

$$Cov\left(\mathbf{z}_{t+3}, \mathbf{z}_t\right) = \mathbf{A}Cov\left(\mathbf{z}_{t+2}, \mathbf{z}_t\right) + \mathbf{B}Cov\left(\boldsymbol{\xi}_{t+3}, \mathbf{z}_t\right)$$

Hence in general

$$Cov\left(\mathbf{z}_{t+s}, \mathbf{z}_t\right) = \mathbf{A}^s Var\left(\mathbf{z}_t\right) + \textstyle\sum_{i=1}^{s} \mathbf{A}^{s-i}\mathbf{B}Cov\left(\boldsymbol{\xi}_{t+i}, \mathbf{z}_t\right)$$

$\Updownarrow$

$$Cov\left(\mathbf{z}_{t+s}, \mathbf{z}_t\right) = \mathbf{A}^s Var\left(\mathbf{z}_t\right) + \textstyle\sum_{j=0}^{s-1} \mathbf{A}^{s-(j+1)}\mathbf{B}Cov\left(\boldsymbol{\xi}_{t+j+1}, \mathbf{z}_t\right)$$

$i = j + 1$ so $j = i - 1$

$\Updownarrow$

$$Cov\left(\mathbf{z}_{t+s}, \mathbf{z}_t\right) = \mathbf{A}^s Var\left(\mathbf{z}_t\right) + \textstyle\sum_{j=0}^{s-1} \mathbf{A}^{s-1-j}\mathbf{B}Cov\left(\boldsymbol{\xi}_{t+j+1}, \mathbf{z}_t\right)$$

or

$$Cov\left(\mathbf{z}_{t+s}, \mathbf{z}_t\right) = \mathbf{A}Cov\left(\mathbf{z}_{t+s-1}, \mathbf{z}_t\right) + \mathbf{B}Cov\left(\boldsymbol{\xi}_{t+s}, \mathbf{z}_t\right)$$

For the control variables:
$Cov\left(\mathbf{y}_{t+s}^{rd}, \mathbf{y}_{t}^{rd}\right) = Cov\left(\mathbf{D}\mathbf{z}_{t+s} + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2, \mathbf{D}\mathbf{z}_{t} + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2\right)$

$$= Cov\left(\mathbf{D}\mathbf{z}_{t+s}, \mathbf{D}\mathbf{z}_{t}\right)$$

$$= \mathbf{D}Cov\left(\mathbf{z}_{t+s}, \mathbf{z}_{t}\right)\mathbf{D}'$$

Thus we only need to compute $Cov\left(\boldsymbol{\xi}_{t+s}, \mathbf{z}_t\right) = E\left[\boldsymbol{\xi}_{t+s}\mathbf{z}_t'\right]$

4.5.3 Computing $Cov\left(\boldsymbol{\xi}_{t+s}, \mathbf{z}_t\right)$

We consider $E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1+s}'\right]$ and note that $E\left[\boldsymbol{\xi}_{t+1+s}\mathbf{z}_t'\right] = \left(E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1+s}'\right]\right)'$.

$$E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1+s}'\right] = E\left[\left[\begin{array}{c} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \\ \mathbf{x}_t^{rd} \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{array}\right]\right.$$

$$\times \left[\begin{array}{l} \boldsymbol{\epsilon}_{t+1+s}' \quad \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s} - vec\left(\mathbf{I}_{n_e}\right)\right)' \quad \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\right)' \\ \left(\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \quad \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^s\right)' \quad \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\otimes\mathbf{x}_{t+s}^f\right)' \\ \left(\mathbf{x}_{t+s}^f\otimes\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \quad \left(\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\right)' \quad \left(\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \quad \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \\ \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\right)' \quad \left(\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right) - E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right]\right)' \end{array}\right]$$

$$= \left[\begin{array}{cccccccccccc} 0_{n_x\times n_e} & 0_{n_x\times n_e^2} & 0_{n_x\times n_e n_x} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_e n_x^2} & 0_{n_x\times n_x^2 n_e} & 0_{n_x\times n_x^2 n_e} & r_{1,9} & r_{1,10} & r_{1,11} & 0_{n_x\times n_e^3} \\ 0_{n_x\times n_e} & 0_{n_x\times n_e^2} & 0_{n_x\times n_e n_x} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_e n_x^2} & 0_{n_x\times n_x^2 n_e} & 0_{n_x\times n_x^2 n_e} & r_{2,9} & r_{2,10} & r_{2,11} & 0_{n_x\times n_e^3} \\ 0_{n_x^2\times n_e} & 0_{n_x^2\times n_e^2} & 0_{n_x^2\times n_e n_x} & 0_{n_x^2\times n_x n_e} & 0_{n_x^2\times n_x n_e} & 0_{n_x^2\times n_e n_x^2} & 0_{n_x^2\times n_x^2 n_e} & 0_{n_x^2\times n_x^2 n_e} & r_{3,9} & r_{3,10} & r_{3,11} & 0_{n_x^2\times n_e^3} \\ 0_{n_x\times n_e} & 0_{n_x\times n_e^2} & 0_{n_x\times n_e n_x} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_e n_x^2} & 0_{n_x\times n_x^2 n_e} & 0_{n_x\times n_x^2 n_e} & r_{4,9} & r_{4,10} & r_{4,11} & 0_{n_x\times n_e^3} \\ 0_{n_x^2\times n_e} & 0_{n_x^2\times n_e^2} & 0_{n_x^2\times n_e n_x} & 0_{n_x^2\times n_x n_e} & 0_{n_x^2\times n_x n_e} & 0_{n_x^2\times n_e n_x^2} & 0_{n_x^2\times n_x^2 n_e} & 0_{n_x^2\times n_x^2 n_e} & r_{5,9} & r_{5,10} & r_{5,1} & 0_{n_x^2\times n_e^3} \\ 0_{n_x^3\times n_e} & 0_{n_x^3\times n_e^2} & 0_{n_x^3\times n_e n_x} & 0_{n_x^3\times n_x n_e} & 0_{n_x^3\times n_x n_e} & 0_{n_x^3\times n_e n_x^2} & 0_{n_x^3\times n_x^2 n_e} & 0_{n_x^3\times n_x^2 n_e} & r_{6,9} & r_{6,10} & r_{6,11} & 0_{n_x^3\times n_e^3} \end{array}\right]$$

$$= \left[\begin{array}{ccc} 0 & \mathbf{R} & 0 \end{array}\right]$$

We now compute the non-zero elements in this matrix

1) The value of $r_{1,9}$
$r_{1,9} = E\left[\mathbf{x}_t^f\left(\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\right]$

$$= E\left[\left\{x_t^f\left(\gamma_1, 1\right)\right\}_{\gamma_1=1}^{n_x}\left\{x_{t+s}^f\left(\gamma_2, 1\right)\left\{\epsilon_{t+1+s}\left(\phi_1, 1\right)\left\{\epsilon_{t+1+s}\left(\phi_2, 1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right]$$

Thus, the quasi Matlab codes are
$E\_xf\_xfeps2 = zeros(nx, nx\times ne\times ne)$

$for\ gama1 = 1 : nx$
  $index2 = 0$
  $for\ gama2 = 1 : nx$
    $for\ phi1 = 1 : ne$
      $for\ phi2 = 1 : ne$
        $index2 = index2 + 1$
        $if\ phi1 == phi2$
          $E\_xf\_xfeps2(gama1, index2) = E\_xf\_xfS(gama1, gama2)$
        end
      end
    end
  end
end
where $E\_xf\_xfS = E\left[\mathbf{x}_t^f\left(\mathbf{x}_{t+s}^f\right)'\right]$

2) The value of $r_{1,10}$
$r_{1,10} = E\left[\mathbf{x}_t^f\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\right]$

$$= E\left[\left\{x_t^f(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1+s}(\phi_1,1)\left\{x_{t+s}^f(\gamma_2,1)\left\{\epsilon_{t+1+s}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right]$$

Thus, the quasi Matlab codes are
$E\_xf\_epsxfeps = zeros(nx, ne\times nx\times ne)$
$for\ gama1 = 1 : nx$
  $index2 = 0$
  $for\ phi1 = 1 : ne$
    $for\ gama2 = 1 : nx$
      $for\ phi2 = 1 : ne$
        $index2 = index2 + 1$
        $if\ phi1 == phi2$
          $E\_xf\_epsxfeps(gama1, index2) = E\_xf\_xfS(gama1, gama2)$
        end
      end
    end
  end
end

3) The value of $r_{1,11}$
$r_{1,11} = E\left[\mathbf{x}_t^f\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\right)'\right]$

$$= E\left[\left\{x_t^f(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1+s}(\phi_1,1)\left\{\epsilon_{t+1+s}(\phi_2,1)\left\{x_{t+s}^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right]$$

Thus, the quasi Matlab codes are
$E\_xf\_eps2xf = zeros(nx, ne\times ne\times nx)$
$for\ gama1 = 1 : nx$
  $index2 = 0$
  $for\ phi1 = 1 : ne$
    $for\ phi2 = 1 : ne$

$for\ gama2 = 1 : nx$
$index2 = index2 + 1$
$if\ phi1 == phi2$
$E\_xf\_eps2xf(gama1, index2) = E\_xf\_xfS(gama1, gama2)$
end
end
end
end
end

4) The value of $r_{2,9}$

$$r_{2,9} = E\left[\mathbf{x}_t^s\left(\mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s}\right)'\right]$$

$$= E\left[\left\{x_t^s\left(\gamma_1,1\right)\right\}_{\gamma_1=1}^{n_x}\left\{x_{t+s}^f\left(\gamma_2,1\right)\left\{\epsilon_{t+1+s}\left(\phi_1,1\right)\left\{\epsilon_{t+1+s}\left(\phi_2,1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right]$$

Thus, the quasi Matlab codes are
$E\_xs\_xfeps2 = zeros(nx, nx \times ne \times ne)$
$for\ gama1 = 1 : nx$
$index2 = 0$
$for\ gama2 = 1 : nx$
$for\ phi1 = 1 : ne$
$for\ phi2 = 1 : ne$
$index2 = index2 + 1$
$if\ phi1 == phi2$
$E\_xs\_xfeps2(gama1, index2) = E\_xs\_xfS(gama1, gama2)$
end
end
end
end
end
where $E\_xs\_xfS = E\left[\mathbf{x}_t^s\left(\mathbf{x}_{t+s}^f\right)'\right]$

5) The value of $r_{2,10}$

$$r_{2,10} = E\left[\mathbf{x}_t^s\left(\boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s}\right)'\right]$$

$$= E\left[\left\{x_t^s\left(\gamma_1,1\right)\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1+s}\left(\phi_1,1\right)\left\{x_{t+s}^f\left(\gamma_2,1\right)\left\{\epsilon_{t+1+s}\left(\phi_2,1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right]$$

Thus, the quasi Matlab codes are
$E\_xs\_epsxfeps = zeros(nx, ne \times nx \times ne)$
$for\ gama1 = 1 : nx$
$index2 = 0$
$for\ phi1 = 1 : ne$
$for\ gama2 = 1 : nx$
$for\ phi2 = 1 : ne$
$index2 = index2 + 1$
$if\ phi1 == phi2$
$E\_xs\_epsxfeps(gama1, index2) = E\_xs\_xfS(gama1, gama2)$

end
end
end
end
end

6) The value of $r_{2,11}$

$$r_{2,11} = E\left[\mathbf{x}_t^s\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\right)'\right]$$

$$= E\left[\left\{x_t^s\left(\gamma_1,1\right)\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1+s}\left(\phi_1,1\right)\left\{\epsilon_{t+1+s}\left(\phi_2,1\right)\left\{x_{t+s}^f\left(\gamma_2,1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right]$$

Thus, the quasi Matlab codes are

```
E_xs_eps2xf = zeros(nx, ne × ne × nx)
for gama1 = 1 : nx
    index2 = 0
    for phi1 = 1 : ne
        for phi2 = 1 : ne
            for gama2 = 1 : nx
                index2 = index2 + 1
                if phi1 == phi2
                    E_xs_eps2xf(gama1, index2) = E_xs_xfS(gama1, gama2)
                end
            end
        end
    end
end
```

7) The value of $r_{3,9}$

$$r_{3,9} = E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\right]$$

$$= E\left[\left\{x_t^f\left(\gamma_1,1\right)\left\{x_t^f\left(\gamma_2,1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{x_{t+s}^f\left(\gamma_3,1\right)\left\{\epsilon_{t+1+s}\left(\phi_1,1\right)\left\{\epsilon_{t+1+s}\left(\phi_2,1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right]$$

Thus, the quasi Matlab codes are

```
E_xfxf_xfeps2 = zeros(nx × nx, nx × ne × ne)
index1 = 0
for gama1 = 1 : nx
    for gama2 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for gama3 = 1 : nx
            for phi1 = 1 : ne
                for phi2 = 1 : ne
                    index2 = index2 + 1
                    if phi1 == phi2
                        E_xfxf_xfeps2(index1, index2) = E_xf_xf_xfS(gama1, gama2, gama3)
                    end
                end
```

end
end
end
end
where $E\_xf\_xf\_xfS = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_{t+s}^f\right)'\right]$

8) The value of $r_{3,10}$

$$r_{3,10} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s}\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1+s}(\phi_1,1)\left\{x_{t+s}^f(\gamma_3,1)\left\{\epsilon_{t+1+s}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right]$$

Thus, the quasi Matlab codes are

$E\_xfxf\_epsxfeps = zeros(nx \times nx, ne \times nx \times ne)$
$index1 = 0$
$for\ gama1 = 1 : nx$
$index1 = index1 + 1$
$for\ gama2 = 1 : nx$
$index2 = 0$
$for\ phi1 = 1 : ne$
$for\ gama3 = 1 : nx$
$for\ phi2 = 1 : ne$
$index2 = index2 + 1$
$if\ phi1 == phi2$
$E\_xfxf\_epsxfeps(index1, index2) = E\_xf\_xf\_xfS(gama1, gama2, gama3)$
end
end
end
end
end
end

9) The value of $r_{3,11}$

$$r_{3,11} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)_{\gamma_2=1}^{n_x}\right\}\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1+s}(\phi_1,1)\left\{\epsilon_{t+1+s}(\phi_2,1)\left\{x_{t+s}^f(\gamma_3,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right]$$

Thus, the quasi Matlab codes are

$E\_xfxf\_eps2xf = zeros(nx \times nx, ne \times ne \times nx)$
$index1 = 0$
$for\ gama1 = 1 : nx$
$for\ gama2 = 1 : nx$
$index1 = index1 + 1$
$index2 = 0$
$for\ phi1 = 1 : ne$
$for\ phi2 = 1 : ne$
$for\ gama3 = 1 : nx$
$index2 = index2 + 1$

$if\ phi1 == phi2$
$E\_xfxf\_eps2xf(index1, index2) = E\_xf\_xf\_xfS(gama1, gama2, gama3)$
end
end
end
end
end
end

10) The value of $r_{4,9}$

$r_{4,9} = E\left[\mathbf{x}_t^{rd}\left(\mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s}\right)'\right]$

$$= E\left[\left\{x_t^{rd}(\gamma_1, 1)\right\}_{\gamma_1=1}^{n_x}\left\{x_{t+s}^f(\gamma_2, 1)\left\{\epsilon_{t+1+s}(\phi_1, 1)\left\{\epsilon_{t+1+s}(\phi_2, 1)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right]$$

Thus, the quasi Matlab codes are

$E\_xrd\_xfeps2 = zeros(nx, nx \times ne \times ne)$
$for\ gama1 = 1 : nx$
$index2 = 0$
$for\ gama2 = 1 : nx$
$for\ phi1 = 1 : ne$
$for\ phi2 = 1 : ne$
$index2 = index2 + 1$
$if\ phi1 == phi2$
$E\_xrd\_xfeps2(gama1, index2) = E\_xrd\_xfS(gama1, gama2)$
end
end
end
end
end

where $E\_xrd\_xfS = E\left[\mathbf{x}_t^{rd}\left(\mathbf{x}_{t+s}^f\right)'\right]$

11) The value of $r_{4,10}$

$r_{4,10} = E\left[\mathbf{x}_t^{rd}\left(\boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s}\right)'\right]$

$$= E\left[\left\{x_t^{rd}(\gamma_1, 1)\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1+s}(\phi_1, 1)\left\{x_{t+s}^f(\gamma_2, 1)\left\{\epsilon_{t+1+s}(\phi_2, 1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right]$$

Thus, the quasi Matlab codes are

$E\_xrd\_epsxfeps = zeros(nx, ne \times nx \times ne)$
$for\ gama1 = 1 : nx$
$index2 = 0$
$for\ phi1 = 1 : ne$
$for\ gama2 = 1 : nx$
$for\ phi2 = 1 : ne$
$index2 = index2 + 1$
$if\ phi1 == phi2$
$E\_xrd\_epsxfeps(gama1, index2) = E\_xrd\_xfS(gama1, gama2)$
end

```
            end
         end
      end
end
```

12) The value of $r_{4,11}$

$$r_{4,11} = E\left[\mathbf{x}_t^{rd}\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^{f}\right)'\right]$$

$$= E\left[\left\{x_t^{rd}(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1+s}(\phi_1,1)\left\{\epsilon_{t+1+s}(\phi_2,1)\left\{x_{t+s}^{f}(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right]$$

Thus, the quasi Matlab codes are

```
E_xrd_eps2xf = zeros(nx, ne × ne × nx)
for gama1 = 1 : nx
   index2 = 0
   for phi1 = 1 : ne
      for phi2 = 1 : ne
         for gama2 = 1 : nx
            index2 = index2 + 1
            if phi1 == phi2
               E_xrd_eps2xf(gama1, index2) = E_xrd_xfS(gama1, gama2)
            end
         end
      end
   end
end
```

13) The value of $r_{5,9}$

$$r_{5,9} = E\left[\left(\mathbf{x}_t^{f}\otimes\mathbf{x}_t^{s}\right)\left(\mathbf{x}_{t+s}^{f}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\right]$$

$$= E\left[\left\{x_t^{f}(\gamma_1,1)\left\{x_t^{s}(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{x_{t+s}^{f}(\gamma_3,1)\left\{\epsilon_{t+1+s}(\phi_1,1)\left\{\epsilon_{t+1+s}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right]$$

Thus, the quasi Matlab codes are

```
E_xfxs_xfeps2 = zeros(nx × nx, nx × ne × ne)
index1 = 0
for gama1 = 1 : nx
   for gama2 = 1 : nx
      index1 = index1 + 1
      index2 = 0
      for gama3 = 1 : nx
         for phi1 = 1 : ne
            for phi2 = 1 : ne
               index2 = index2 + 1
               if phi1 == phi2
                  E_xfxs_xfeps2(index1, index2) = E_xf_xs_xfS(gama1, gama2, gama3)
               end
            end
         end
```

end
end
end
where $E\_xf\_xs\_xfS = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\left(\mathbf{x}_{t+s}^f\right)'\right]$

14) The value of $r_{5,10}$

$$r_{5,10} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s}\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^s(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1+s}(\phi_1,1)\left\{x_{t+s}^f(\gamma_3,1)\left\{\epsilon_{t+1+s}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right]$$

Thus, the quasi Matlab codes are

```
E_xfxs_epsxfeps = zeros(nx × nx, ne × nx × ne)
index1 = 0
for gama1 = 1 : nx
    for gama2 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for phi1 = 1 : ne
            for gama3 = 1 : nx
                for phi2 = 1 : ne
                    index2 = index2 + 1
                    if phi1 == phi2
                        E_xfxs_epsxfeps(index1, index2) = E_xf_xs_xfS(gama1, gama2, gama3)
                    end
                end
            end
        end
    end
end
```

15) The value of $r_{5,11}$

$$r_{5,11} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^s(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1+s}(\phi_1,1)\left\{\epsilon_{t+1+s}(\phi_2,1)\left\{x_{t+s}^f(\gamma_3,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right]$$

Thus, the quasi Matlab codes are

```
E_xfxs_eps2xf = zeros(nx × nx, ne × ne × nx)
index1 = 0
for gama1 = 1 : nx
    for gama2 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for phi1 = 1 : ne
            for phi2 = 1 : ne
                for gama3 = 1 : nx
                    index2 = index2 + 1
                    if phi1 == phi2
```

$E\_xfxs\_eps2xf(index1, index2) = E\_xf\_xs\_xfS(gama1, gama2, gama3)$
end
end
end
end
end
end

16) The value of $r_{6,9}$

$$r_{6,9} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s}\right)'\right]$$

$$= E[\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\left\{x_t^f(\gamma_3,1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x} \times \left\{x_{t+s}^f(\gamma_3,1)\left\{\epsilon_{t+1+s}(\phi_1,1)\left\{\epsilon_{t+1+s}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}]$$

Thus, the quasi Matlab codes are

$E\_xfxfxf\_xfeps2 = zeros(nx \times nx \times nx, nx \times ne \times ne)$
$index1 = 0$
$for\ gama1 = 1 : nx$
$for\ gama2 = 1 : nx$
$for\ gama3 = 1 : nx$
$index1 = index1 + 1$
$index2 = 0$
$for\ gama4 = 1 : nx$
$for\ phi1 = 1 : ne$
$for\ phi2 = 1 : ne$
$index2 = index2 + 1$
$if\ phi1 == phi2$
$E\_xfxfxf\_xfeps2(index1, index2)$
$= E\_xf\_xf\_xf\_xfS(gama1, gama2, gama3, gama4)$
end
end
end
end
end
end
end

where $E\_xf\_xf\_xf\_xfS = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_{t+s}^f\right)'\right]$

17) The value of $r_{6,10}$

$$r_{6,10} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s}\right)'\right]$$

$$= E[\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\left\{x_t^f(\gamma_3,1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x} \times \left\{\epsilon_{t+1+s}(\phi_1,1)\left\{x_{t+s}^f(\gamma_3,1)\left\{\epsilon_{t+1+s}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_1=1}^{n_e}]$$

Thus, the quasi Matlab codes are

$E\_xfxfxf\_epsxfeps = zeros(nx \times nx \times nx, ne \times nx \times ne)$
$index1 = 0$
$for\ gama1 = 1 : nx$
$\quad for\ gama2 = 1 : nx$
$\quad\quad for\ gama3 = 1 : nx$
$\quad\quad\quad index1 = index1 + 1$
$\quad\quad\quad index2 = 0$
$\quad\quad\quad for\ phi1 = 1 : ne$
$\quad\quad\quad\quad for\ gama4 = 1 : nx$
$\quad\quad\quad\quad\quad for\ phi2 = 1 : ne$
$\quad\quad\quad\quad\quad\quad index2 = index2 + 1$
$\quad\quad\quad\quad\quad\quad if\ phi1 == phi2$
$\quad\quad\quad\quad\quad\quad\quad E\_xfxfxf\_epsxfeps(index1, index2)$
$\quad\quad\quad\quad\quad\quad\quad = E\_xf\_xf\_xf\_xfS(gama1, gama2, gama3, gama4)$
$\quad\quad\quad\quad\quad\quad end$
$\quad\quad\quad\quad\quad end$
$\quad\quad\quad\quad end$
$\quad\quad\quad end$
$\quad\quad end$
$\quad end$
end

18) The value of $r_{6,11}$

$$r_{6,11} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f\right)'\right]$$

$$= E[\left\{x_t^f(\gamma_1, 1)\left\{x_t^f(\gamma_2, 1)\left\{x_t^f(\gamma_3, 1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}$$
$$\times\left\{\epsilon_{t+1+s}(\phi_1, 1)\left\{\epsilon_{t+1+s}(\phi_2, 1)\left\{x_{t+s}^f(\gamma_3, 1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}]$$

Thus, the quasi Matlab codes are

$E\_xfxfxf\_eps2xf = zeros(nx \times nx \times nx, ne \times ne \times nx)$
$index1 = 0$
$for\ gama1 = 1 : nx$
$\quad for\ gama2 = 1 : nx$
$\quad\quad for\ gama3 = 1 : nx$
$\quad\quad\quad index1 = index1 + 1$
$\quad\quad\quad index2 = 0$
$\quad\quad\quad for\ phi1 = 1 : ne$
$\quad\quad\quad\quad for\ phi2 = 1 : ne$
$\quad\quad\quad\quad\quad for\ gama4 = 1 : nx$
$\quad\quad\quad\quad\quad\quad index2 = index2 + 1$
$\quad\quad\quad\quad\quad\quad if\ phi1 == phi2$
$\quad\quad\quad\quad\quad\quad\quad E\_xfxfxf\_eps2xf(index1, index2)$
$\quad\quad\quad\quad\quad\quad\quad = E\_xf\_xf\_xf\_xfS(gama1, gama2, gama3, gama4)$
$\quad\quad\quad\quad\quad\quad end$
$\quad\quad\quad\quad\quad end$
$\quad\quad\quad\quad end$
$\quad\quad\quad end$

end

end

end

We know all the required moments, except $E\left[\mathbf{x}_t^f\left(\mathbf{x}_{t+s}^f\right)'\right]$, $E\left[\mathbf{x}_t^s\left(\mathbf{x}_{t+s}^f\right)'\right]$, $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_{t+s}^f\right)'\right]$, $E\left[\mathbf{x}_t^{rd}\left(\mathbf{x}_{t+s}^f\right)'\right]$, $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_{t+s}^f\right)'\right]$, and $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_{t+s}^f\right)'\right]$

a) For $E\left[\mathbf{x}_t^f\left(\mathbf{x}_{t+s}^f\right)'\right]$

Recall that $\mathbf{x}_{t+s}^f=\mathbf{h}_\mathbf{x}^s\mathbf{x}_t^f+\sum_{i=1}^s\mathbf{h}_\mathbf{x}^{s-i}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+i}$.

So

$E\left[\mathbf{x}_t^f\left(\mathbf{x}_{t+s}^f\right)'\right]=E\left[\mathbf{x}_t^f\left(\mathbf{h}_\mathbf{x}^s\mathbf{x}_t^f+\sum_{i=1}^s\mathbf{h}_\mathbf{x}^{s-i}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+i}\right)'\right]=E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\left(\mathbf{h}_\mathbf{x}^s\right)'$

b) For $E\left[\mathbf{x}_t^s\left(\mathbf{x}_{t+s}^f\right)'\right]$

$E\left[\mathbf{x}_t^s\left(\mathbf{x}_{t+s}^f\right)'\right]=E\left[\mathbf{x}_t^s\left(\mathbf{h}_\mathbf{x}^s\mathbf{x}_t^f+\sum_{i=1}^s\mathbf{h}_\mathbf{x}^{s-i}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+i}\right)'\right]=E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\right)'\right]\left(\mathbf{h}_\mathbf{x}^s\right)'$

c) For $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_{t+s}^f\right)'\right]$

$E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_{t+s}^f\right)'\right]=E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{h}_\mathbf{x}^s\mathbf{x}_t^f+\sum_{i=1}^s\mathbf{h}_\mathbf{x}^{s-i}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+i}\right)'\right]=E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right]\left(\mathbf{h}_\mathbf{x}^s\right)'$

d) For $E\left[\mathbf{x}_t^{rd}\left(\mathbf{x}_{t+s}^f\right)'\right]$

$E\left[\mathbf{x}_t^{rd}\left(\mathbf{x}_{t+s}^f\right)'\right]=E\left[\mathbf{x}_t^{rd}\left(\mathbf{h}_\mathbf{x}^s\mathbf{x}_t^f+\sum_{i=1}^s\mathbf{h}_\mathbf{x}^{s-i}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+i}\right)'\right]$

$=E\left[\mathbf{x}_t^{rd}\left(\mathbf{h}_\mathbf{x}^s\mathbf{x}_t^f\right)'\right]$

$=E\left[\mathbf{x}_t^{rd}\left(\mathbf{x}_t^f\right)'\right]\left(\mathbf{h}_\mathbf{x}^s\right)'$

So we only need to find $E\left[\mathbf{x}_t^{rd}\left(\mathbf{x}_t^f\right)'\right]$. Recall that

$\mathbf{x}_{t+1}^{rd}=\mathbf{h}_\mathbf{x}\mathbf{x}_t^{rd}+2\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)+\tilde{\mathbf{H}}_{\mathbf{xxx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$

So

$E\left[\mathbf{x}_t^{rd}\left(\mathbf{x}_t^f\right)'\right]$

$=E\left[\left(\mathbf{h}_\mathbf{x}\mathbf{x}_t^{rd}+2\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)+\tilde{\mathbf{H}}_{\mathbf{xxx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\right)\left(\mathbf{x}_t^f\right)'\right]$

$=\mathbf{h}_\mathbf{x}E\left[\mathbf{x}_t^{rd}\left(\mathbf{x}_t^f\right)'\right]+2\tilde{\mathbf{H}}_{\mathbf{xx}}E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\right)'\right]$

$+\tilde{\mathbf{H}}_{\mathbf{xxx}}E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right]+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]$

$\Updownarrow$

$$E\left[\mathbf{x}_t^{rd}\left(\mathbf{x}_t^f\right)'\right] = (\mathbf{I}-\mathbf{h_x})^{-1}\left[2\tilde{\mathbf{H}}_{\mathbf{xx}}E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\right)'\right]+\tilde{\mathbf{H}}_{\mathbf{xxx}}E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right]+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\right]$$

e) For $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_{t+s}^f\right)'\right]$

$$E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_{t+s}^f\right)'\right]$$

$$= E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\left(\mathbf{h}_\mathbf{x}^s\mathbf{x}_t^f+\sum_{i=1}^s\mathbf{h}_\mathbf{x}^{s-i}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+i}\right)'\right]$$

$$= E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\right)'\right](\mathbf{h}_\mathbf{x}^s)'$$

f) For $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_{t+s}^f\right)'\right]$

$$E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_{t+s}^f\right)'\right]$$

$$= E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{h}_\mathbf{x}^s\mathbf{x}_t^f+\sum_{i=1}^s\mathbf{h}_\mathbf{x}^{s-i}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+i}\right)'\right]$$

$$= E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right](\mathbf{h}_\mathbf{x}^s)'$$

5 The Dynare++ notation

This section presents the pruning method up to third order using the notation in Dynare and Dyanare++. The solution to DSGE models are in Dynare and Dynare++ given by

$$\mathbf{z}_t = \mathbf{f}\left(\mathbf{z}_{t-1},\mathbf{u}_t,\sigma\right) \tag{39}$$

where $\mathbf{z}_t$ contains all the endogenous variables (i.e. control variables and all state variables), and $\mathbf{u}_t$ with size $n_u\times 1$ is the vector of disturbances with the property $\mathbf{u}_t\sim\mathcal{NID}\left(\mathbf{0},\boldsymbol{\Sigma}\right)$. It is convenient to express this more general solution in a notation that is similar to the one used above. We therefore write (39) as

$$\mathbf{y}_t = \mathbf{g}\left(\mathbf{x}_{t-1},\mathbf{u}_t,\sigma\right) \tag{40}$$

$$\mathbf{x}_{t+1} = \mathbf{h}\left(\mathbf{x}_t,\mathbf{u}_{t+1},\sigma\right) \tag{41}$$

where $\mathbf{y}_t$ and $\mathbf{x}_t$ are as defined above. The key difference compared to the notation in Schmitt-Grohé & Uribe (2004) is that the function $\mathbf{g}$ depends on the innovations $\mathbf{u}_t$. Note also that the innovations may enter in a non-linear fashion in the $\mathbf{h}$ function. Below, it is useful to define

$$\mathbf{v}_{t,t+1}\equiv\begin{bmatrix}\mathbf{x}_t\\ \mathbf{u}_{t+1}\end{bmatrix} \tag{42}$$

where $\mathbf{v}_{t,t+1}$ has dimensions $n_v\times 1$. The first subscribt of $\mathbf{v}_{t,t+1}$ refers to the time index of $\mathbf{x}_t$ and the second to the time index of $\mathbf{u}_{t+1}$.

A first-order approximation (40) and (41) around the deterministic steady state is

$$\mathbf{y}_t = \mathbf{g}_\mathbf{v}\mathbf{v}_{t-1,t} \tag{43}$$

$$\mathbf{x}_{t+1} = \mathbf{h}_{\mathbf{v}}\mathbf{v}_{t,t+1} \tag{44}$$

A second-order approximation is

$$\mathbf{y}_t = \mathbf{g}_{\mathbf{v}}\mathbf{v}_{t-1,t} + \frac{1}{2}\mathbf{G}_{\mathbf{vv}}\left(\mathbf{v}_{t-1,t} \otimes \mathbf{v}_{t-1,t}\right) + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 \tag{45}$$

and

$$\mathbf{x}_{t+1} = \mathbf{h}_{\mathbf{v}}\mathbf{v}_{t,t+1} + \frac{1}{2}\mathbf{H}_{\mathbf{vv}}\left(\mathbf{v}_{t,t+1} \otimes \mathbf{v}_{t,t+1}\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \tag{46}$$

A third-order approximation is

$$\begin{aligned} \mathbf{y}_t &= \mathbf{g}_{\mathbf{v}}\mathbf{v}_{t-1,t} + \frac{1}{2}\mathbf{G}_{\mathbf{vv}}\left(\mathbf{v}_{t-1,t} \otimes \mathbf{v}_{t-1,t}\right) + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 \\ &+\frac{1}{6}\mathbf{G}_{\mathbf{vvv}}\left(\mathbf{v}_{t-1,t} \otimes \mathbf{v}_{t-1,t} \otimes \mathbf{v}_{t-1,t}\right) + \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{v}}\sigma^2\mathbf{v}_{t-1,t} + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3 \end{aligned} \tag{47}$$

and

$$\begin{aligned} \mathbf{x}_{t+1} &= \mathbf{h}_{\mathbf{v}}\mathbf{v}_{t,t+1} + \frac{1}{2}\mathbf{H}_{\mathbf{vv}}\left(\mathbf{v}_{t,t+1} \otimes \mathbf{v}_{t,t+1}\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\ &+\frac{1}{2}\mathbf{H}_{\mathbf{vvv}}\left(\mathbf{v}_{t,t+1} \otimes \mathbf{v}_{t,t+1} \otimes \mathbf{v}_{t,t+1}\right) + \frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{v}}\sigma^2\mathbf{v}_{t,t+1} + \frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3 \end{aligned} \tag{48}$$

6 Pruning scheme in Dynare++:

6.1 Second order approximation:

We start considering (46) which we write as

$$x_{t+1}(j,1) = \mathbf{h}_{\mathbf{v}}(j,:)\,\mathbf{v}_{t,t+1} + \left(\mathbf{v}_{t,t+1}\right)'\mathbf{h}_{\mathbf{vv}}(j,:,:)\,\mathbf{v}_{t,t+1} + \tfrac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2$$

$$\Updownarrow$$

$$x_{t+1}(j,1) = \mathbf{h}_{\mathbf{v}}(j,:)\begin{bmatrix}\mathbf{x}_t \\ \mathbf{u}_{t+1}\end{bmatrix} + \begin{bmatrix}\mathbf{x}_t \\ \mathbf{u}_{t+1}\end{bmatrix}'\mathbf{h}_{\mathbf{vv}}(j,:,:)\begin{bmatrix}\mathbf{x}_t \\ \mathbf{u}_{t+1}\end{bmatrix} + \tfrac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2$$

for $j = 1, 2, ..., n_x$.

Let us now decompose the state vector $\mathbf{x}_t$ as

$$\mathbf{x}_t = \mathbf{x}_t^f + \mathbf{x}_t^s \tag{49}$$

Notice, that we do not need to compose the innovations as they are a first order effect in the system. For the subsequent decomposition let

$$\mathbf{h}_{\mathbf{v}} \equiv \begin{bmatrix}\mathbf{h}_{\mathbf{x}}(j,:) & \mathbf{h}_{\mathbf{u}}(j,:)\end{bmatrix} \tag{50}$$

$$\mathbf{h}_{\mathbf{vv}}(j,:,:) = \begin{bmatrix}\mathbf{h}_{\mathbf{xx}}(j,:,:) & \mathbf{h}_{\mathbf{xu}}(j,:,:) \\ \mathbf{h}_{\mathbf{ux}}(j,:,:) & \mathbf{h}_{\mathbf{uu}}(j,:,:)\end{bmatrix} \tag{51}$$

for $j = 1, 2, ..., n_x$.

Hence,

$$x_{t+1}^f(j,1) + x_{t+1}^s(j,1) = \begin{bmatrix}\mathbf{h}_{\mathbf{x}}(j,:) & \mathbf{h}_{\mathbf{u}}(j,:)\end{bmatrix}\begin{bmatrix}\mathbf{x}_t^f + \mathbf{x}_t^s \\ \mathbf{u}_{t+1}\end{bmatrix}$$

$$+\tfrac{1}{2}\begin{bmatrix}\mathbf{x}_t^f + \mathbf{x}_t^s \\ \mathbf{u}_{t+1}\end{bmatrix}'\begin{bmatrix}\mathbf{h}_{\mathbf{xx}}(j,:,:) & \mathbf{h}_{\mathbf{xu}}(j,:,:) \\ \mathbf{h}_{\mathbf{ux}}(j,:,:) & \mathbf{h}_{\mathbf{uu}}(j,:,:)\end{bmatrix}\begin{bmatrix}\mathbf{x}_t^f + \mathbf{x}_t^s \\ \mathbf{u}_{t+1}\end{bmatrix} + \tfrac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2$$

$$\Updownarrow$$

$$x^f_{t+1}(j,1)+x^s_{t+1}(j,1)=\mathbf{h_x}(j,:)\left(\mathbf{x}^f_t+\mathbf{x}^s_t\right)+\mathbf{h_u}(j,:)\mathbf{u}_{t+1}$$
$$+\tfrac{1}{2}\left[\begin{array}{cc}\left(\mathbf{x}^f_t\right)'+(\mathbf{x}^s_t)' & \mathbf{u}'_{t+1}\end{array}\right]\left[\begin{array}{cc}\mathbf{h_{xx}}(j,:,:) & \mathbf{h_{xu}}(j,:,:)\\ \mathbf{h_{ux}}(j,:,:) & \mathbf{h_{uu}}(j,:,:)\end{array}\right]\left[\begin{array}{c}\mathbf{x}^f_t+\mathbf{x}^s_t\\ \mathbf{u}_{t+1}\end{array}\right]$$
$$+\tfrac{1}{2}h_{\sigma\sigma}(j,1)\sigma^2$$
$\Updownarrow$

$$x^f_{t+1}(j,1)+x^s_{t+1}(j,1)=\mathbf{h_x}(j,:)\left(\mathbf{x}^f_t+\mathbf{x}^s_t\right)+\mathbf{h_u}(j,:)\mathbf{u}_{t+1}$$
$$+\tfrac{1}{2}\left[\begin{array}{cc}\left(\left(\mathbf{x}^f_t\right)'+(\mathbf{x}^s_t)'\right)\mathbf{h_{xx}}(j,:,:)+\mathbf{u}'_{t+1}\mathbf{h_{ux}}(j,:,:) & \left(\left(\mathbf{x}^f_t\right)'+(\mathbf{x}^s_t)'\right)\mathbf{h_{xu}}(j,:,:)+\mathbf{u}'_{t+1}\mathbf{h_{uu}}(j,:,:)\end{array}\right]\left[\begin{array}{c}\mathbf{x}^f_t+\mathbf{x}^s_t\\ \mathbf{u}_{t+1}\end{array}\right]$$
$$+\tfrac{1}{2}h_{\sigma\sigma}(j,1)\sigma^2$$
$\Updownarrow$

$$x^f_{t+1}(j,1)+x^s_{t+1}(j,1)=\mathbf{h_x}(j,:)\left(\mathbf{x}^f_t+\mathbf{x}^s_t\right)+\mathbf{h_u}(j,:)\mathbf{u}_{t+1}$$
$$+\tfrac{1}{2}\left(\left(\mathbf{x}^f_t\right)'+(\mathbf{x}^s_t)'\right)\mathbf{h_{xx}}(j,:,:)\left(\mathbf{x}^f_t+\mathbf{x}^s_t\right)+\tfrac{1}{2}\mathbf{u}'_{t+1}\mathbf{h_{ux}}(j,:,:)\left(\mathbf{x}^f_t+\mathbf{x}^s_t\right)$$
$$+\tfrac{1}{2}\left(\left(\mathbf{x}^f_t\right)'+(\mathbf{x}^s_t)'\right)\mathbf{h_{xu}}(j,:,:)\mathbf{u}_{t+1}+\tfrac{1}{2}\mathbf{u}'_{t+1}\mathbf{h_{uu}}(j,:,:)\mathbf{u}_{t+1}$$
$$+\tfrac{1}{2}h_{\sigma\sigma}(j,1)\sigma^2$$

A law of motion for the first-order terms is thus

$$x^f_{t+1}(j,1)=\mathbf{h_x}(j,:)\mathbf{x}^f_t+\mathbf{h_u}(j,:)\mathbf{u}_{t+1} \quad (52)$$

for $j=1,2,...,n_x$.

A law of motion for the second-order terms is thus

$$\begin{aligned} x^s_{t+1}(j,1) &= \mathbf{h_x}(j,:)\mathbf{x}^s_t+\frac{1}{2}\left(\mathbf{x}^f_t\right)'\mathbf{h_{xx}}(j,:,:)\left(\mathbf{x}^f_t\right)+\frac{1}{2}\mathbf{u}'_{t+1}\mathbf{h_{ux}}(j,:,:)\mathbf{x}^f_t \\ &\quad +\frac{1}{2}\left(\mathbf{x}^f_t\right)'\mathbf{h_{xu}}(j,:,:)\mathbf{u}_{t+1}+\frac{1}{2}\mathbf{u}'_{t+1}\mathbf{h_{uu}}(j,:,:)\mathbf{u}_{t+1}+\frac{1}{2}h_{\sigma\sigma}(j,1)\sigma^2 \end{aligned} \quad (53)$$

for $j=1,2,...,n_x$. Note here, that non-linear shocks will imply that we have innovations to $x^s_{t+1}(j,1)$, i.e. if $\mathbf{h_{ux}}(j,:,:)\neq\mathbf{0}$ and $\mathbf{h_{uu}}(j,:,:)\neq\mathbf{0}$.

For the control variables, we introduce the following notation

$$\mathbf{g_v}\equiv\left[\begin{array}{cc}\mathbf{g_x}(i,:) & \mathbf{g_u}(i,:)\end{array}\right] \quad (54)$$

$$\mathbf{g_{vv}}(i,:,:)=\left[\begin{array}{cc}\mathbf{g_{xx}}(i,:,:) & \mathbf{g_{xu}}(i,:,:)\\ \mathbf{g_{ux}}(i,:,:) & \mathbf{g_{uu}}(i,:,:)\end{array}\right] \quad (55)$$

for $i=1,2,...,n_y$. Thus
$y_t(i,1)=\mathbf{g_v}(i,:)\mathbf{v}_{t-1,t}+\frac{1}{2}(\mathbf{v}_{t-1,t})'\mathbf{g_{vv}}(i,:,:)\mathbf{v}_{t-1,t}+\frac{1}{2}\mathbf{g}_{\sigma\sigma}(i,1)\sigma^2$
$\Updownarrow$

$$y_t(i,1)=\left[\begin{array}{cc}\mathbf{g_x}(i,:) & \mathbf{g_u}(i,:)\end{array}\right]\left[\begin{array}{c}\mathbf{x}^f_{t-1}+\mathbf{x}^s_{t-1}\\ \mathbf{u}_t\end{array}\right]$$
$$+\tfrac{1}{2}\left[\begin{array}{c}\mathbf{x}^f_{t-1}+\mathbf{x}^s_{t-1}\\ \mathbf{u}_t\end{array}\right]'\left[\begin{array}{cc}\mathbf{g_{xx}}(i,:,:) & \mathbf{g_{xu}}(i,:,:)\\ \mathbf{g_{ux}}(i,:,:) & \mathbf{g_{uu}}(i,:,:)\end{array}\right]\left[\begin{array}{c}\mathbf{x}^f_{t-1}+\mathbf{x}^s_{t-1}\\ \mathbf{u}_t\end{array}\right]+\tfrac{1}{2}g_{\sigma\sigma}(i,1)\sigma^2$$
$\Updownarrow$

$$y_t(i,1) = \mathbf{g_x}(i,:)\left(\mathbf{x}_{t-1}^f + \mathbf{x}_{t-1}^s\right) + \mathbf{g_u}(i,:)\mathbf{u}_t$$
$$+\tfrac{1}{2}\begin{bmatrix} \left(\mathbf{x}_{t-1}^f\right)' + \left(\mathbf{x}_{t-1}^s\right)' & \mathbf{u}_t' \end{bmatrix} \begin{bmatrix} \mathbf{g_{xx}}(i,:,:) & \mathbf{g_{xu}}(i,:,:) \\ \mathbf{g_{ux}}(i,:,:) & \mathbf{g_{uu}}(i,:,:) \end{bmatrix} \begin{bmatrix} \mathbf{x}_{t-1}^f + \mathbf{x}_{t-1}^s \\ \mathbf{u}_t \end{bmatrix} + \tfrac{1}{2} g_{\sigma\sigma}(i,1)\sigma^2$$
$$\Updownarrow$$
$$y_t(i,1) = \mathbf{g_x}(i,:)\left(\mathbf{x}_{t-1}^f + \mathbf{x}_{t-1}^s\right) + \mathbf{g_u}(i,:)\mathbf{u}_t$$
$$+\tfrac{1}{2}\begin{bmatrix} \left(\left(\mathbf{x}_{t-1}^f\right)' + \left(\mathbf{x}_{t-1}^s\right)'\right)\mathbf{g_{xx}}(i,:,:) + \mathbf{u}_t'\mathbf{g_{ux}}(i,:,:) & \left(\left(\mathbf{x}_{t-1}^f\right)' + \left(\mathbf{x}_{t-1}^s\right)'\right)\mathbf{g_{xu}}(i,:,:) + \mathbf{u}_t'\mathbf{g_{uu}}(i,:,:) \end{bmatrix}$$
$$\times \begin{bmatrix} \mathbf{x}_{t-1}^f + \mathbf{x}_{t-1}^s \\ \mathbf{u}_t \end{bmatrix}$$
$$+\tfrac{1}{2} g_{\sigma\sigma}(i,1)\sigma^2$$
$$\Updownarrow$$
$$y_t(i,1) = \mathbf{g_x}(i,:)\left(\mathbf{x}_{t-1}^f + \mathbf{x}_{t-1}^s\right) + \mathbf{g_u}(i,:)\mathbf{u}_t$$
$$+\tfrac{1}{2}\left(\left(\mathbf{x}_{t-1}^f\right)' + \left(\mathbf{x}_{t-1}^s\right)'\right)\mathbf{g_{xx}}(i,:,:)\left(\mathbf{x}_{t-1}^f + \mathbf{x}_{t-1}^s\right) + \tfrac{1}{2}\mathbf{u}_t'\mathbf{g_{ux}}(i,:,:)\left(\mathbf{x}_{t-1}^f + \mathbf{x}_{t-1}^s\right)$$
$$+\tfrac{1}{2}\left(\left(\mathbf{x}_{t-1}^f\right)' + \left(\mathbf{x}_{t-1}^s\right)'\right)\mathbf{g_{xu}}(i,:,:)\mathbf{u}_t + \tfrac{1}{2}\mathbf{u}_t'\mathbf{g_{uu}}(i,:,:)\mathbf{u}_t$$
$$+\tfrac{1}{2} g_{\sigma\sigma}(i,1)\sigma^2$$

for $i = 1, 2, ..., n_y$. We want to preserve terms up to second order, hence the pruned approximation is

$$\begin{aligned} y_t(i,1) = \; & \mathbf{g_x}(i,:)\left(\mathbf{x}_{t-1}^f + \mathbf{x}_{t-1}^s\right) + \mathbf{g_u}(i,:)\mathbf{u}_t \\ & +\frac{1}{2}\left(\mathbf{x}_{t-1}^f\right)'\mathbf{g_{xx}}(i,:,:)\mathbf{x}_{t-1}^f + \frac{1}{2}\mathbf{u}_t'\mathbf{g_{ux}}(i,:,:)\mathbf{x}_{t-1}^f \\ & +\frac{1}{2}\left(\mathbf{x}_{t-1}^f\right)'\mathbf{g_{xu}}(i,:,:)\mathbf{u}_t + \frac{1}{2}\mathbf{u}_t'\mathbf{g_{uu}}(i,:,:)\mathbf{u}_t \\ & +\frac{1}{2} g_{\sigma\sigma}(i,1)\sigma^2 \end{aligned} \tag{56}$$

for $i = 1, 2, ..., n_y$.

6.2 Second order approximation: a convenient representation

When coding the derived formulas it is convenient to use $\mathbf{v}_{t,t+1}$ directly, and the corresponding derivatives of $\mathbf{g}$ and $\mathbf{h}$, because this is how the output from Dynare and Dynare++ is stored. Hence, we can write

$$\mathbf{x}_{t+1}^f = \mathbf{h_v}\begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_{t+1} \end{bmatrix} \tag{57}$$

and

$$x_{t+1}^s(j,:) = \mathbf{h_v}(j,:)\begin{bmatrix} \mathbf{x}_t^s \\ \mathbf{0} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{u}_{t+1} \end{bmatrix}'\mathbf{h_{vv}}(j,:,:)\begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{u}_{t+1} \end{bmatrix} + \frac{1}{2}h_{\sigma\sigma}(j,:)\sigma^2 \tag{58}$$

for $j = 1, 2, ..., n_x$. For the control variables we have

$$y_t(i,1) = \mathbf{g_v}(i,:)\left(\begin{bmatrix} \mathbf{x}_{t-1}^f \\ \mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \mathbf{x}_{t-1}^s \\ \mathbf{0} \end{bmatrix}\right) + \frac{1}{2}\begin{bmatrix} \mathbf{x}_{t-1}^f \\ \mathbf{u}_t \end{bmatrix}'\mathbf{g_{vv}}(i,:,:)\begin{bmatrix} \mathbf{x}_{t-1}^f \\ \mathbf{u}_t \end{bmatrix} + \frac{1}{2}g_{\sigma\sigma}(i,1)\sigma^2 \tag{59}$$

for $i = 1, 2, ..., n_y$.

Using the kronecker representation (fast for MATLAB) we have

$$\mathbf{x}_{t+1}^{s}=\mathbf{h}_{\mathbf{v}}\left[\begin{array}{c}\mathbf{x}_{t}^{s}\\ \mathbf{0}\end{array}\right]+\tilde{\mathbf{H}}_{\mathbf{vv}}\left(\left[\begin{array}{c}\mathbf{x}_{t}^{f}\\ \mathbf{u}_{t+1}\end{array}\right]\otimes\left[\begin{array}{c}\mathbf{x}_{t}^{f}\\ \mathbf{u}_{t+1}\end{array}\right]\right)+\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2} \tag{60}$$

where

$$\tilde{\mathbf{H}}_{\mathbf{vv}}\equiv\frac{1}{2}reshape\left(\mathbf{h}_{\mathbf{vv}},n_{x},n_{v}^{2}\right) \tag{61}$$

And

$$\mathbf{y}_{t}=\mathbf{g}_{\mathbf{v}}\left(\left[\begin{array}{c}\mathbf{x}_{t-1}^{f}\\ \mathbf{u}_{t}\end{array}\right]+\left[\begin{array}{c}\mathbf{x}_{t-1}^{s}\\ \mathbf{0}\end{array}\right]\right)+\tilde{\mathbf{G}}_{\mathbf{vv}}\left(\left[\begin{array}{c}\mathbf{x}_{t-1}^{f}\\ \mathbf{u}_{t}\end{array}\right]\otimes\left[\begin{array}{c}\mathbf{x}_{t-1}^{f}\\ \mathbf{u}_{t}\end{array}\right]\right)+\frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^{2} \tag{62}$$

where

$$\tilde{\mathbf{G}}_{\mathbf{vv}}\equiv\frac{1}{2}reshape\left(\mathbf{g}_{\mathbf{vv}},n_{y},n_{v}^{2}\right) \tag{63}$$

6.3 Third order approximation:

Let us now decompose the state vector $\mathbf{x}_t$ as

$$\mathbf{x}_{t}=\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}+\mathbf{x}_{t}^{rd} \tag{64}$$

Thus

$$x_{t+1}(j,1)=\mathbf{h}_{\mathbf{v}}(j,:)\,\mathbf{v}_{t,t+1}+\tfrac{1}{2}\left(\mathbf{v}_{t,t+1}\right)'\mathbf{h}_{\mathbf{vv}}(j,:,:)\,\mathbf{v}_{t,t+1}+\tfrac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^{2}$$
$$+\tfrac{1}{6}\left(\mathbf{v}_{t,t+1}\right)'\left[\begin{array}{c}\left(\mathbf{v}_{t,t+1}\right)'\mathbf{h}_{\mathbf{vvv}}(j,1,:,:)\mathbf{v}_{t,t+1}\\ \ldots\\ \left(\mathbf{v}_{t,t+1}\right)'\mathbf{h}_{\mathbf{vvv}}(j,n_{v},:,:)\mathbf{v}_{t,t+1}\end{array}\right]+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{v}}(j,:)\,\sigma^{2}\mathbf{v}_{t,t+1}+\tfrac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^{3}$$

$\Updownarrow$

$$x_{t+1}^{f}(j,1)+x_{t+1}^{s}(j,1)+x_{t+1}^{rd}(j,1)=\left[\begin{array}{cc}\mathbf{h}_{\mathbf{x}}(j,:) & \mathbf{h}_{\mathbf{u}}(j,:)\end{array}\right]\left[\begin{array}{c}\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}+\mathbf{x}_{t}^{rd}\\ \mathbf{u}_{t+1}\end{array}\right]$$
$$+\tfrac{1}{2}\left[\begin{array}{c}\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}+\mathbf{x}_{t}^{rd}\\ \mathbf{u}_{t+1}\end{array}\right]'\left[\begin{array}{cc}\mathbf{h}_{\mathbf{xx}}(j,:,:) & \mathbf{h}_{\mathbf{xu}}(j,:,:)\\ \mathbf{h}_{\mathbf{ux}}(j,:,:) & \mathbf{h}_{\mathbf{uu}}(j,:,:)\end{array}\right]\left[\begin{array}{c}\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}+\mathbf{x}_{t}^{rd}\\ \mathbf{u}_{t+1}\end{array}\right]+\tfrac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^{2}$$
$$+\tfrac{1}{6}\left[\begin{array}{cc}\left(\mathbf{x}_{t}^{f}\right)'+\left(\mathbf{x}_{t}^{s}\right)'+\left(\mathbf{x}_{t}^{rd}\right)' & \mathbf{u}_{t+1}\end{array}\right]\left[\begin{array}{c}\left[\begin{array}{cc}\left(\mathbf{x}_{t}^{f}\right)'+\left(\mathbf{x}_{t}^{s}\right)'+\left(\mathbf{x}_{t}^{rd}\right)' & \mathbf{u}_{t+1}\end{array}\right]\mathbf{h}_{\mathbf{vvv}}(j,1,:,:)\left[\begin{array}{c}\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}+\mathbf{x}_{t}^{rd}\\ \mathbf{u}_{t+1}\end{array}\right]\\ \ldots\\ \left[\begin{array}{cc}\left(\mathbf{x}_{t}^{f}\right)'+\left(\mathbf{x}_{t}^{s}\right)'+\left(\mathbf{x}_{t}^{rd}\right)' & \mathbf{u}_{t+1}\end{array}\right]\mathbf{h}_{\mathbf{vvv}}(j,n_{v},:,:)\left[\begin{array}{c}\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}+\mathbf{x}_{t}^{rd}\\ \mathbf{u}_{t+1}\end{array}\right]\end{array}\right]$$
$$+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{v}}(j,:)\,\sigma^{2}\left[\begin{array}{c}\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}+\mathbf{x}_{t}^{rd}\\ \mathbf{u}_{t+1}\end{array}\right]+\tfrac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^{3}$$

$\Updownarrow$

$$x_{t+1}^{f}(j,1)+x_{t+1}^{s}(j,1)+x_{t+1}^{rd}(j,1)=\mathbf{h}_{\mathbf{x}}(j,:)\left(\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}+\mathbf{x}_{t}^{rd}\right)+\mathbf{h}_{\mathbf{u}}(j,:)\,\mathbf{u}_{t+1}$$
$$+\tfrac{1}{2}\left(\left(\mathbf{x}_{t}^{f}\right)'+\left(\mathbf{x}_{t}^{s}\right)'+\left(\mathbf{x}_{t}^{rd}\right)'\right)\mathbf{h}_{\mathbf{xx}}(j,:,:)\left(\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}+\mathbf{x}_{t}^{rd}\right)+\tfrac{1}{2}\mathbf{u}_{t+1}'\mathbf{h}_{\mathbf{ux}}(j,:,:)\left(\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}+\mathbf{x}_{t}^{rd}\right)$$
$$+\tfrac{1}{2}\left(\left(\mathbf{x}_{t}^{f}\right)'+\left(\mathbf{x}_{t}^{s}\right)'+\left(\mathbf{x}_{t}^{rd}\right)'\right)\mathbf{h}_{\mathbf{xu}}(j,:,:)\,\mathbf{u}_{t+1}+\tfrac{1}{2}\mathbf{u}_{t+1}'\mathbf{h}_{\mathbf{uu}}(j,:,:)\,\mathbf{u}_{t+1}$$
$$+\tfrac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^{2}$$

$$+\tfrac{1}{6}\left[\begin{array}{cc}\left(\mathbf{x}_t^f\right)'+(\mathbf{x}_t^s)'+\left(\mathbf{x}_t^{rd}\right)' & \mathbf{u}_{t+1}\end{array}\right]\left[\begin{array}{c}\left[\begin{array}{cc}\left(\mathbf{x}_t^f\right)'+(\mathbf{x}_t^s)'+\left(\mathbf{x}_t^{rd}\right)' & \mathbf{u}_{t+1}\end{array}\right]\mathbf{h_{vvv}}(j,1,:,:)\left[\begin{array}{c}\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\\ \mathbf{u}_{t+1}\end{array}\right]\\ \ldots\\ \left[\begin{array}{cc}\left(\mathbf{x}_t^f\right)'+(\mathbf{x}_t^s)'+\left(\mathbf{x}_t^{rd}\right)' & \mathbf{u}_{t+1}\end{array}\right]\mathbf{h_{vvv}}(j,n_v,:,:)\left[\begin{array}{c}\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\\ \mathbf{u}_{t+1}\end{array}\right]\end{array}\right]$$
$$+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{v}}(j,:)\,\sigma^2\left[\begin{array}{c}\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\\ \mathbf{u}_{t+1}\end{array}\right]+\tfrac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3$$

Without reducing the large term for with $\mathbf{h_{vvv}}(j,:,:,:)$, it is straightforward to see that the law of motion for $x_{t+1}^{rd}(j,1)$ that only preserves third order terms is

$$x_{t+1}^{rd}(j,1)=\mathbf{h_x}(j,:)\,\mathbf{x}_t^{rd}+$$
$$+\tfrac{1}{2}(\mathbf{x}_t^s)'\,\mathbf{h_{xx}}(j,:,:)\,\mathbf{x}_t^f+\tfrac{1}{2}\left(\mathbf{x}_t^f\right)'\mathbf{h_{xx}}(j,:,:)\,\mathbf{x}_t^s+\tfrac{1}{2}\mathbf{u}_{t+1}'\mathbf{h_{ux}}(j,:,:)\,\mathbf{x}_t^s$$
$$+\tfrac{1}{2}(\mathbf{x}_t^s)'\,\mathbf{h_{xu}}(j,:,:)\,\mathbf{u}_{t+1}$$
$$+\tfrac{1}{6}\left[\begin{array}{cc}\left(\mathbf{x}_t^f\right)' & \mathbf{u}_{t+1}\end{array}\right]\left[\begin{array}{c}\left[\begin{array}{cc}\left(\mathbf{x}_t^f\right)' & \mathbf{u}_{t+1}\end{array}\right]\mathbf{h_{vvv}}(j,1,:,:)\left[\begin{array}{c}\mathbf{x}_t^f\\ \mathbf{u}_{t+1}\end{array}\right]\\ \ldots\\ \left[\begin{array}{cc}\left(\mathbf{x}_t^f\right)' & \mathbf{u}_{t+1}\end{array}\right]\mathbf{h_{vvv}}(j,n_v,:,:)\left[\begin{array}{c}\mathbf{x}_t^f\\ \mathbf{u}_{t+1}\end{array}\right]\end{array}\right]$$
$$+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\tilde{\mathbf{x}}}(j,:)\,\sigma^2\left[\begin{array}{c}\mathbf{x}_t^f\\ \mathbf{u}_{t+1}\end{array}\right]+\tfrac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3$$

Using the convenient representation we thus have

$$\begin{aligned}\mathbf{x}_{t+1}^{rd} &= \mathbf{h_v}\left[\begin{array}{c}\mathbf{x}_t^{rd}\\ \mathbf{0}\end{array}\right]\\ &\quad+2\tilde{\mathbf{H}}_{\mathbf{vv}}\left(\left[\begin{array}{c}\mathbf{x}_t^f\\ \mathbf{u}_{t+1}\end{array}\right]\otimes\left[\begin{array}{c}\mathbf{x}_t^s\\ \mathbf{0}\end{array}\right]\right)\\ &\quad+\tilde{\mathbf{H}}_{\mathbf{vvv}}\left(\left[\begin{array}{c}\mathbf{x}_t^f\\ \mathbf{u}_{t+1}\end{array}\right]\otimes\left[\begin{array}{c}\mathbf{x}_t^f\\ \mathbf{u}_{t+1}\end{array}\right]\otimes\left[\begin{array}{c}\mathbf{x}_t^f\\ \mathbf{u}_{t+1}\end{array}\right]\right)\\ &\quad+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{v}}\sigma^2\left[\begin{array}{c}\mathbf{x}_t^f\\ \mathbf{u}_{t+1}\end{array}\right]+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^2\end{aligned}\tag{65}$$

It is by now straigthforward to see that an expression for $\mathbf{y}_t$ which only preserves up to third order terms are:

$$\begin{aligned}\mathbf{y}_t &= \mathbf{g_v}\left(\left[\begin{array}{c}\mathbf{x}_{t-1}^f\\ \mathbf{u}_t\end{array}\right]+\left[\begin{array}{c}\mathbf{x}_{t-1}^s\\ \mathbf{0}\end{array}\right]+\left[\begin{array}{c}\mathbf{x}_{t-1}^{rd}\\ \mathbf{0}\end{array}\right]\right)\\ &\quad+\tilde{\mathbf{G}}_{\mathbf{vv}}\left(\left(\left[\begin{array}{c}\mathbf{x}_{t-1}^f\\ \mathbf{u}_t\end{array}\right]\otimes\left[\begin{array}{c}\mathbf{x}_{t-1}^f\\ \mathbf{u}_t\end{array}\right]\right)+2\left(\left[\begin{array}{c}\mathbf{x}_{t-1}^f\\ \mathbf{u}_t\end{array}\right]\otimes\left[\begin{array}{c}\mathbf{x}_{t-1}^s\\ \mathbf{0}\end{array}\right]\right)\right)\\ &\quad+\tilde{\mathbf{G}}_{\mathbf{vvv}}\left(\left[\begin{array}{c}\mathbf{x}_{t-1}^f\\ \mathbf{u}_t\end{array}\right]\otimes\left[\begin{array}{c}\mathbf{x}_{t-1}^f\\ \mathbf{u}_t\end{array}\right]\otimes\left[\begin{array}{c}\mathbf{x}_{t-1}^f\\ \mathbf{u}_t\end{array}\right]\right)\\ &\quad+\frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2+\frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{v}}\sigma^2\left[\begin{array}{c}\mathbf{x}_{t-1}^f\\ \mathbf{u}_t\end{array}\right]+\frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^2\end{aligned}\tag{66}$$

7 Dynare++ notation and statistical properties: second order

7.1 Co-variance stationarity

We start with the state variables. From above we have for the first-order effects that

$$x^f_{t+1}(j,1) = \mathbf{h_x}(j,:)\,\mathbf{x}^f_t + \mathbf{h_u}(j,:)\,\mathbf{u}_{t+1}$$
$$\Downarrow$$
$$\mathbf{x}^f_{t+1} = \mathbf{h_x}\mathbf{x}^f_t + \mathbf{h_u}\mathbf{u}_{t+1}$$

For the second-order effects

$$x^s_{t+1}(j,1) = \mathbf{h_x}(j,:)\,\mathbf{x}^s_t + \tfrac{1}{2}\left(\mathbf{x}^f_t\right)'\mathbf{h_{xx}}(j,:,:)\left(\mathbf{x}^f_t\right) + \tfrac{1}{2}\mathbf{u}'_{t+1}\mathbf{h_{ux}}(j,:,:)\,\mathbf{x}^f_t$$
$$+\tfrac{1}{2}\left(\mathbf{x}^f_t\right)'\mathbf{h_{xu}}(j,:,:)\,\mathbf{u}_{t+1} + \mathbf{u}'_{t+1}\mathbf{h_{uu}}(j,:,:)\,\mathbf{u}_{t+1} + \tfrac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2$$
$$\Downarrow$$
$$\mathbf{x}^s_{t+1} = \mathbf{h_x}\mathbf{x}^s_t + \tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right) + \mathbf{u}'_{t+1}\mathbf{h_{ux}}(j,:,:)\left(\mathbf{x}^f_t\right) + \left(\mathbf{x}^f_t\right)'\mathbf{h_{xu}}(j,:,:)\,\mathbf{u}_{t+1} + \tilde{\mathbf{H}}_{\mathbf{uu}}\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$
$$\Updownarrow$$
$$\mathbf{x}^s_{t+1} = \mathbf{h_x}\mathbf{x}^s_t + \tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right) + \tilde{\mathbf{H}}_{\mathbf{ux}}\left(\mathbf{u}_{t+1}\otimes\mathbf{x}^f_t\right) + \tilde{\mathbf{H}}_{\mathbf{xu}}\left(\mathbf{x}^f_t\otimes\mathbf{u}_{t+1}\right) + \tilde{\mathbf{H}}_{\mathbf{uu}}\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$
$$\Updownarrow$$
$$\mathbf{x}^s_{t+1} = \mathbf{h_x}\mathbf{x}^s_t + \tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right) + \tilde{\mathbf{H}}_{\mathbf{ux}}\left(\mathbf{u}_{t+1}\otimes\mathbf{x}^f_t\right) + \tilde{\mathbf{H}}_{\mathbf{xu}}\left(\mathbf{x}^f_t\otimes\mathbf{u}_{t+1}\right) + \tilde{\mathbf{H}}_{\mathbf{uu}}\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1} - vec(\mathbf{\Sigma}) + vec(\mathbf{\Sigma})\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$
$$\Updownarrow$$
$$\mathbf{x}^s_{t+1} = \mathbf{h_x}\mathbf{x}^s_t + \tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right) + \tilde{\mathbf{H}}_{\mathbf{ux}}\left(\mathbf{u}_{t+1}\otimes\mathbf{x}^f_t\right) + \tilde{\mathbf{H}}_{\mathbf{xu}}\left(\mathbf{x}^f_t\otimes\mathbf{u}_{t+1}\right) + \tilde{\mathbf{H}}_{\mathbf{uu}}\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1} - vec(\mathbf{\Sigma})\right)$$
$$+\tilde{\mathbf{H}}_{\mathbf{uu}}vec(\mathbf{\Sigma}) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$

where we have defined

$$\tilde{\mathbf{H}}_{\mathbf{xx}} \equiv \tfrac{1}{2}reshape\left(\mathbf{h_{xx}}, n_x, n_x^2\right)$$
$$\tilde{\mathbf{H}}_{\mathbf{uu}} \equiv \tfrac{1}{2}reshape\left(\mathbf{h_{uu}}, n_x, n_u^2\right)$$
$$\tilde{\mathbf{H}}_{\mathbf{ux}} \equiv \tfrac{1}{2}reshape\left(\mathbf{h_{ux}}, n_x, n_u n_x\right)$$
$$\tilde{\mathbf{H}}_{\mathbf{xu}} \equiv \tfrac{1}{2}reshape\left(\mathbf{h_{xu}}, n_x, n_x n_u\right)$$

Hence, we need to find the law of motions for $\mathbf{x}^f_t\otimes\mathbf{x}^f_t$

$$\mathbf{x}^f_t\otimes\mathbf{x}^f_t = \left(\mathbf{h_x}\mathbf{x}^f_{t-1} + \mathbf{h_u}\mathbf{u}_t\right)\otimes\left(\mathbf{h_x}\mathbf{x}^f_{t-1} + \mathbf{h_u}\mathbf{u}_t\right)$$

$$= \mathbf{h_x}\mathbf{x}^f_{t-1}\otimes\left(\mathbf{h_x}\mathbf{x}^f_{t-1} + \mathbf{h_u}\mathbf{u}_t\right) + \mathbf{h_u}\mathbf{u}_t\otimes\left(\mathbf{h_x}\mathbf{x}^f_{t-1} + \mathbf{h_u}\mathbf{u}_t\right)$$

$$= \mathbf{h_x}\mathbf{x}^f_{t-1}\otimes\mathbf{h_x}\mathbf{x}^f_{t-1} + \mathbf{h_x}\mathbf{x}^f_{t-1}\otimes\mathbf{h_u}\mathbf{u}_t + \mathbf{h_u}\mathbf{u}_t\otimes\mathbf{h_x}\mathbf{x}^f_{t-1} + \mathbf{h_u}\mathbf{u}_t\otimes\mathbf{h_u}\mathbf{u}_t$$

$$= \left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{x}^f_{t-1}\otimes\mathbf{x}^f_{t-1}\right) + \left(\mathbf{h_x}\otimes\mathbf{h_u}\right)\left(\mathbf{x}^f_{t-1}\otimes\mathbf{u}_t\right)$$
$$+\left(\mathbf{h_u}\otimes\mathbf{h_x}\right)\left(\mathbf{u}_t\otimes\mathbf{x}^f_{t-1}\right) + \left(\mathbf{h_u}\otimes\mathbf{h_u}\right)\left(\mathbf{u}_t\otimes\mathbf{u}_t\right)$$

$$= \left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{x}^f_{t-1}\otimes\mathbf{x}^f_{t-1}\right) + \left(\mathbf{h_x}\otimes\mathbf{h_u}\right)\left(\mathbf{x}^f_{t-1}\otimes\mathbf{u}_t\right)$$
$$+\left(\mathbf{h_u}\otimes\mathbf{h_x}\right)\left(\mathbf{u}_t\otimes\mathbf{x}^f_{t-1}\right) + \left(\mathbf{h_u}\otimes\mathbf{h_u}\right)\left(\mathbf{u}_t\otimes\mathbf{u}_t\right)$$

Thus

$$\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1} = \left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right) + \left(\mathbf{h_x}\otimes\mathbf{h_u}\right)\left(\mathbf{x}^f_t\otimes\mathbf{u}_{t+1}\right)$$
$$+\left(\mathbf{h_u}\otimes\mathbf{h_x}\right)\left(\mathbf{u}_{t+1}\otimes\mathbf{x}^f_t\right) + \left(\mathbf{h_u}\otimes\mathbf{h_u}\right)\left(\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}\right) - vec(\mathbf{\Sigma}) + vec(\mathbf{\Sigma})\right)$$

$$
\begin{aligned}
&= (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\mathbf{h_x} \otimes \mathbf{h_u}) \left(\mathbf{x}_t^f \otimes \mathbf{u}_{t+1}\right) \\
&\quad + (\mathbf{h_u} \otimes \mathbf{h_x}) \left(\mathbf{u}_{t+1} \otimes \mathbf{x}_t^f\right) + (\mathbf{h_u} \otimes \mathbf{h_u}) \left((\mathbf{u}_{t+1} \otimes \mathbf{u}_{t+1}) - vec\,(\mathbf{\Sigma})\right) \\
&\quad + (\mathbf{h_u} \otimes \mathbf{h_u})\, vec\,(\mathbf{\Sigma})
\end{aligned}
$$

Thus we can set up the following system

$$
\begin{bmatrix} \mathbf{x}_{t+1}^f \\ \mathbf{x}_{t+1}^s \\ \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \tilde{\mathbf{H}}_{\mathbf{uu}} vec\,(\mathbf{\Sigma}) \\ (\mathbf{h_u} \otimes \mathbf{h_u})\, vec\,(\mathbf{\Sigma}) \end{bmatrix} + \begin{bmatrix} \mathbf{h_x} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{h_x} & \tilde{\mathbf{H}}_{\mathbf{xx}} \\ \mathbf{0} & \mathbf{0} & \mathbf{h_x} \otimes \mathbf{h_x} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{bmatrix}
$$
$$
+ \begin{bmatrix} \mathbf{h_u} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{H}}_{\mathbf{uu}} & \tilde{\mathbf{H}}_{\mathbf{ux}} & \tilde{\mathbf{H}}_{\mathbf{xu}} \\ \mathbf{0} & \mathbf{h_u} \otimes \mathbf{h_u} & (\mathbf{h_u} \otimes \mathbf{h_x}) & (\mathbf{h_u} \otimes \mathbf{h_u}) \end{bmatrix} \begin{bmatrix} \mathbf{u}_{t+1} \\ \mathbf{u}_{t+1} \otimes \mathbf{u}_{t+1} - vec\,(\mathbf{\Sigma}) \\ \mathbf{u}_{t+1} \otimes \mathbf{x}_t^f \\ \mathbf{x}_t^f \otimes \mathbf{u}_{t+1} \end{bmatrix}
$$

$\Updownarrow$

$$
\mathbf{z}_{t+1} = \mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}
$$

where we have defined

$$
\mathbf{c} \equiv \begin{bmatrix} \mathbf{0} \\ \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \tilde{\mathbf{H}}_{\mathbf{uu}} vec\,(\mathbf{\Sigma}) \\ (\mathbf{h_u} \otimes \mathbf{h_u})\, vec\,(\mathbf{\Sigma}) \end{bmatrix}
$$

$$
\mathbf{A} \equiv \begin{bmatrix} \mathbf{h_x} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{h_x} & \tilde{\mathbf{H}}_{\mathbf{xx}} \\ \mathbf{0} & \mathbf{0} & \mathbf{h_x} \otimes \mathbf{h_x} \end{bmatrix}
$$

$$
\mathbf{B} \equiv \begin{bmatrix} \mathbf{h_u} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{H}}_{\mathbf{uu}} & \tilde{\mathbf{H}}_{\mathbf{ux}} & \tilde{\mathbf{H}}_{\mathbf{xu}} \\ \mathbf{0} & \mathbf{h_u} \otimes \mathbf{h_u} & (\mathbf{h_u} \otimes \mathbf{h_x}) & (\mathbf{h_u} \otimes \mathbf{h_u}) \end{bmatrix}
$$

$$
\boldsymbol{\xi}_{t+1} \equiv \begin{bmatrix} \mathbf{u}_{t+1} \\ \mathbf{u}_{t+1} \otimes \mathbf{u}_{t+1} - vec\,(\mathbf{\Sigma}) \\ \mathbf{u}_{t+1} \otimes \mathbf{x}_t^f \\ \mathbf{x}_t^f \otimes \mathbf{u}_{t+1} \end{bmatrix}
$$

Similar arguments as presented above ensure that all eigenvalues of $\mathbf{A}$ have modulus less than one provided the same holds for $\mathbf{h_x}$.

For the control variables we have

$$
\begin{aligned}
&y_t\,(i,1) = \mathbf{g_x}\,(i,:) \left(\mathbf{x}_{t-1}^f + \mathbf{x}_{t-1}^s\right) + \mathbf{g_u}\,(i,:)\,\mathbf{u}_t \\
&\quad + \tfrac{1}{2} \left(\mathbf{x}_{t-1}^f\right)' \mathbf{g_{xx}}\,(i,:,:)\,\mathbf{x}_{t-1}^f + \tfrac{1}{2}\mathbf{u}_t'\mathbf{g_{ux}}\,(i,:,:)\,\mathbf{x}_{t-1}^f \\
&\quad + \tfrac{1}{2} \left(\mathbf{x}_{t-1}^f\right)' \mathbf{g_{xu}}\,(i,:,:)\,\mathbf{u}_t + \tfrac{1}{2}\mathbf{u}_t'\mathbf{g_{uu}}\,(i,:,:)\,\mathbf{u}_t \\
&\quad + \tfrac{1}{2} g_{\sigma\sigma}\,(i,1)\,\sigma^2
\end{aligned}
$$

$\Downarrow$

$$
\begin{aligned}
\mathbf{y}_t &= \mathbf{g_x} \left(\mathbf{x}_{t-1}^f + \mathbf{x}_{t-1}^s\right) + \mathbf{g_u}\mathbf{u}_t + \tilde{\mathbf{G}}_{\mathbf{xx}} \left(\mathbf{x}_{t-1}^f \otimes \mathbf{x}_{t-1}^f\right) \\
&\quad + \tilde{\mathbf{G}}_{\mathbf{ux}} \left(\mathbf{u}_t \otimes \mathbf{x}_{t-1}^f\right) + \tilde{\mathbf{G}}_{\mathbf{xu}} \left(\mathbf{x}_{t-1}^f \otimes \mathbf{u}_t\right) + \tilde{\mathbf{G}}_{\mathbf{uu}} \left(\mathbf{u}_t \otimes \mathbf{u}_t\right) + \tfrac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 \\
&= \mathbf{g_x} \left(\mathbf{x}_{t-1}^f + \mathbf{x}_{t-1}^s\right) + \mathbf{g_u}\mathbf{u}_t + \tilde{\mathbf{G}}_{\mathbf{xx}} \left(\mathbf{x}_{t-1}^f \otimes \mathbf{x}_{t-1}^f\right)
\end{aligned}
$$

$$+\tilde{\mathbf{G}}_{\mathbf{ux}}\left(\mathbf{u}_t \otimes \mathbf{x}_{t-1}^f\right) + \tilde{\mathbf{G}}_{\mathbf{xu}}\left(\mathbf{x}_{t-1}^f \otimes \mathbf{u}_t\right) + \tilde{\mathbf{G}}_{\mathbf{uu}}\left((\mathbf{u}_t \otimes \mathbf{u}_t) - vec(\mathbf{\Sigma})\right)$$
$$+\frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \tilde{\mathbf{G}}_{\mathbf{uu}} vec(\mathbf{\Sigma})$$

where we have defined

$\tilde{\mathbf{G}}_{\mathbf{xx}} \equiv \frac{1}{2} reshape\left(\mathbf{g}_{\mathbf{xx}}, n_y, n_x^2\right)$
$\tilde{\mathbf{G}}_{\mathbf{uu}} \equiv \frac{1}{2} reshape\left(\mathbf{g}_{\mathbf{uu}}, n_y, n_u^2\right)$
$\tilde{\mathbf{G}}_{\mathbf{ux}} \equiv \frac{1}{2} reshape\left(\mathbf{g}_{\mathbf{ux}}, n_y, n_u n_x\right)$
$\tilde{\mathbf{G}}_{\mathbf{xu}} \equiv \frac{1}{2} reshape\left(\mathbf{g}_{\mathbf{xu}}, n_y, n_x n_u\right)$

Thus

$$\mathbf{y}_t = \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \tilde{\mathbf{G}}_{\mathbf{uu}} vec(\mathbf{\Sigma}) + \begin{bmatrix} \mathbf{g}_{\mathbf{x}} & \mathbf{g}_{\mathbf{x}} & \tilde{\mathbf{G}}_{\mathbf{xx}} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{t-1}^f \\ \mathbf{x}_{t-1}^s \\ \mathbf{x}_{t-1}^f \otimes \mathbf{x}_{t-1}^f \end{bmatrix}$$
$$+ \begin{bmatrix} \mathbf{g}_{\mathbf{u}} & \tilde{\mathbf{G}}_{\mathbf{uu}} & \tilde{\mathbf{G}}_{\mathbf{ux}} & \tilde{\mathbf{G}}_{\mathbf{xu}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_t \\ \mathbf{u}_t \otimes \mathbf{u}_t - vec(\mathbf{\Sigma}) \\ \mathbf{u}_t \otimes \mathbf{x}_{t-1}^f \\ \mathbf{x}_{t-1}^f \otimes \mathbf{u}_t \end{bmatrix}$$

$\Updownarrow$

$$\mathbf{y}_t = \mathbf{d} + \mathbf{E}\mathbf{z}_t + \mathbf{F}\boldsymbol{\xi}_t$$

7.2 First and second moments

We also see that $E_t\left[\boldsymbol{\xi}_{t+1}\right] = \mathbf{0}$. Hence, the first and second moments for $\mathbf{z}_t$ are:

$$E[\mathbf{z}_t] = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{c}$$

and for the variances we have that

$$E\left[\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right] = E\left[\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}\right)\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}\right)'\right]$$

$$= E\left[\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}\right)\left(\mathbf{c}' + \mathbf{z}_t'\mathbf{A}' + \boldsymbol{\xi}_{t+1}'\mathbf{B}'\right)\right]$$

$$= E\left[\mathbf{c}\left(\mathbf{c}' + \mathbf{z}_t'\mathbf{A}' + \boldsymbol{\xi}_{t+1}'\mathbf{B}'\right)\right]$$
$$+E\left[\mathbf{A}\mathbf{z}_t\left(\mathbf{c}' + \mathbf{z}_t'\mathbf{A}' + \boldsymbol{\xi}_{t+1}'\mathbf{B}'\right)\right]$$
$$+E\left[\mathbf{B}\boldsymbol{\xi}_{t+1}\left(\mathbf{c}' + \mathbf{z}_t'\mathbf{A}' + \boldsymbol{\xi}_{t+1}'\mathbf{B}'\right)\right]$$

$$= E\left[\mathbf{c}\mathbf{c}' + \mathbf{c}\mathbf{z}_t'\mathbf{A}' + \mathbf{c}\boldsymbol{\xi}_{t+1}'\mathbf{B}'\right]$$
$$+E\left[\mathbf{A}\mathbf{z}_t\mathbf{c}' + \mathbf{A}\mathbf{z}_t\mathbf{z}_t'\mathbf{A}' + \mathbf{A}\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\mathbf{B}'\right]$$
$$+E\left[\mathbf{B}\boldsymbol{\xi}_{t+1}\mathbf{c}' + \mathbf{B}\boldsymbol{\xi}_{t+1}\mathbf{z}_t'\mathbf{A}' + \mathbf{B}\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\mathbf{B}'\right]$$

$$= \mathbf{c}\mathbf{c}' + \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}'$$
$$+\mathbf{A}E[\mathbf{z}_t]\mathbf{c}' + \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{A}E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$
$$+\mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$

We then note that

$$E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\right] = E\left[\begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{bmatrix}\begin{bmatrix} \mathbf{u}_{t+1}' & \left(\mathbf{u}_{t+1} \otimes \mathbf{u}_{t+1} - vec(\mathbf{\Sigma})\right)' & \left(\mathbf{u}_{t+1} \otimes \mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{u}_{t+1}\right)' \end{bmatrix}\right]$$

$$= E\begin{bmatrix} \mathbf{x}_t^f \mathbf{u}_{t+1}' & \mathbf{x}_t^f \left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1} - vec\left(\mathbf{\Sigma}\right)\right)' & \mathbf{x}_t^f \left(\mathbf{u}_{t+1}\otimes\mathbf{x}_t^f\right)' & \mathbf{x}_t^f \left(\mathbf{x}_t^f\otimes\mathbf{u}_{t+1}\right)' \\ \mathbf{x}_t^s \mathbf{u}_{t+1}' & \mathbf{x}_t^s \left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1} - vec\left(\mathbf{\Sigma}\right)\right)' & \mathbf{x}_t^s \left(\mathbf{u}_{t+1}\otimes\mathbf{x}_t^f\right)' & \mathbf{x}_t^s \left(\mathbf{x}_t^f\otimes\mathbf{u}_{t+1}\right)' \\ \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\mathbf{u}_{t+1}' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1} - vec\left(\mathbf{\Sigma}\right)\right)' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{u}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{u}_{t+1}\right)' \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Thus

$$E\left[\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right] = \mathbf{c}\mathbf{c}' + \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{A}E\left[\mathbf{z}_t\right]\mathbf{c}' + \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$

$$= \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' + \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' + \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$

Note also that

$$E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t\right]' = \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)\left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)'$$

$$= \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' + \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)E\left[\mathbf{z}_t'\right]\mathbf{A}'$$

$$= \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' + \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{A}E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\mathbf{A}'$$

So

$$E\left[\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right] - E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t\right]' = \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' + \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' + \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$
$$- \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' - \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' - \mathbf{A}E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\mathbf{A}'$$

$$= \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}' - \mathbf{A}E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\mathbf{A}'$$

$$= \mathbf{A}\left(E\left[\mathbf{z}_t\mathbf{z}_t'\right] - E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\right)\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$

$\Updownarrow$

$$Var\left(\mathbf{z}_t\right) = \mathbf{A}Var\left(\mathbf{z}_t\right)\mathbf{A}' + \mathbf{B}Var\left(\boldsymbol{\xi}_{t+1}\right)\mathbf{B}'$$

For the control we have directly that

$$E\left[\mathbf{y}_t\right] = \mathbf{d} + \mathbf{E}E\left[\mathbf{z}_t\right]$$

$$Var\left[\mathbf{y}_t\right] = \mathbf{E}Var\left[\mathbf{z}_t\right]\mathbf{E}' + \mathbf{F}Var\left(\boldsymbol{\xi}_t\right)\mathbf{F}'$$

where we use that $Cov\left(\mathbf{z}_t, \boldsymbol{\xi}_t\right) = 0$. Note that we trivially have $E\left[\mathbf{x}_t^f\right] = \mathbf{0}$ which we will use below. Hence, we only need to compute $Var\left(\boldsymbol{\xi}_{t+1}\right)$.

7.2.1 Computing $Var\left(\boldsymbol{\xi}_{t+1}\right)$

We have

$$Var\left(\boldsymbol{\xi}_{t+1}\right) = E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]$$

$$= E\left[\begin{bmatrix} \mathbf{u}_{t+1} \\ \mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1} - vec\left(\mathbf{\Sigma}\right) \\ \mathbf{u}_{t+1}\otimes\mathbf{x}_t^f \\ \mathbf{x}_t^f\otimes\mathbf{u}_{t+1} \end{bmatrix}\begin{matrix} \mathbf{u}_{t+1}' & \left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}\right)' - vec\left(\mathbf{I}\right)' & \left(\mathbf{u}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f\otimes\mathbf{u}_{t+1}\right)' \end{matrix}\right]$$

$$
= E \left[\begin{array}{cc}
\mathbf{u}_{t+1}\mathbf{u}'_{t+1} & \mathbf{u}_{t+1}\left(\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}\right)-vec\left(\mathbf{\Sigma}\right)\right)' \\
\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}-vec\left(\mathbf{\Sigma}\right)\right)\mathbf{u}'_{t+1} & \left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}-vec\left(\mathbf{I}\right)\right)\left(\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}\right)-vec\left(\mathbf{\Sigma}\right)\right)' \\
\left(\mathbf{u}_{t+1}\otimes\mathbf{x}^f_t\right)\mathbf{u}'_{t+1} & \left(\mathbf{u}_{t+1}\otimes\mathbf{x}^f_t\right)\left(\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}\right)-vec\left(\mathbf{\Sigma}\right)\right)' \\
\left(\mathbf{x}^f_t\otimes\mathbf{u}_{t+1}\right)\mathbf{u}'_{t+1} & \left(\mathbf{x}^f_t\otimes\mathbf{u}_{t+1}\right)\left(\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}\right)-vec\left(\mathbf{\Sigma}\right)\right)'
\end{array}\right.
$$

$$
\left. \begin{array}{cc}
\mathbf{u}_{t+1}\left(\mathbf{u}_{t+1}\otimes\mathbf{x}^f_t\right)' & \mathbf{u}_{t+1}\left(\mathbf{x}^f_t\otimes\mathbf{u}_{t+1}\right)' \\
\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}-vec\left(\mathbf{\Sigma}\right)\right)\left(\mathbf{u}_{t+1}\otimes\mathbf{x}^f_t\right)' & \left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}-vec\left(\mathbf{\Sigma}\right)\right)\left(\mathbf{x}^f_t\otimes\mathbf{u}_{t+1}\right)' \\
\left(\mathbf{u}_{t+1}\otimes\mathbf{x}^f_t\right)\left(\mathbf{u}_{t+1}\otimes\mathbf{x}^f_t\right)' & \left(\mathbf{u}_{t+1}\otimes\mathbf{x}^f_t\right)\left(\mathbf{x}^f_t\otimes\mathbf{u}_{t+1}\right)' \\
\left(\mathbf{x}^f_t\otimes\mathbf{u}_{t+1}\right)\left(\mathbf{u}_{t+1}\otimes\mathbf{x}^f_t\right)' & \left(\mathbf{x}^f_t\otimes\mathbf{u}_{t+1}\right)\left(\mathbf{x}^f_t\otimes\mathbf{u}_{t+1}\right)'
\end{array}\right]
$$

$$
= \left[\begin{array}{cc}
\mathbf{\Sigma} & E\left[\mathbf{u}_{t+1}\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}\right)'\right] \\
E\left[\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}\right)\mathbf{u}'_{t+1}\right] & E\left[\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}-vec\left(\mathbf{\Sigma}\right)\right)\left(\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}\right)-vec\left(\mathbf{\Sigma}\right)\right)'\right] \\
\mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0}
\end{array}\right.
$$

$$
\left. \begin{array}{cc}
\mathbf{0} & \mathbf{0}' \\
\mathbf{0} & \mathbf{0} \\
\left(\mathbf{u}_{t+1}\otimes\mathbf{x}^f_t\right)\left(\mathbf{u}_{t+1}\otimes\mathbf{x}^f_t\right)' & \left(\mathbf{u}_{t+1}\otimes\mathbf{x}^f_t\right)\left(\mathbf{x}^f_t\otimes\mathbf{u}_{t+1}\right)' \\
\left(\mathbf{x}^f_t\otimes\mathbf{u}_{t+1}\right)\left(\mathbf{u}_{t+1}\otimes\mathbf{x}^f_t\right)' & \left(\mathbf{x}^f_t\otimes\mathbf{u}_{t+1}\right)\left(\mathbf{x}^f_t\otimes\mathbf{u}_{t+1}\right)'
\end{array}\right]
$$

These elements can be coded directly as shown above.

8 Equivalence between the SGU-notation and the Dynare notation

This section shows the equivalence between the notation by (Schmitt-Grohé & Uribe (2004)), i.e. the SGU-notation, where innovations only enter linearly and the Dynare and Dynare ++ notation where innovations may enter in a non-linear fashion. The key observation is that the SGU-notation actually also includes the Dynare and Dynare ++ notation when extending the state vector accordingly. Recall from above that the SGU-notation reads:

$$\mathbf{y}_t = \mathbf{g}\left(\mathbf{x}_t, \sigma\right) \tag{67}$$

$$\mathbf{x}_{t+1} = \mathbf{h}\left(\mathbf{x}_t, \sigma\right) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \tag{68}$$

By lagging (68) by one period we get

$$\mathbf{x}_t = \mathbf{h}\left(\mathbf{x}_{t-1}, \sigma\right) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_t \tag{69}$$

and $\mathbf{v}_t \equiv \left[\begin{array}{cc}\mathbf{x}_{t-1} & \boldsymbol{\epsilon}_t\end{array}\right]$ can then be considered as the extended state vector .Hence, for this extended system we thus have

$$\mathbf{y}_t = \mathbf{g}\left(\mathbf{x}_{t-1}, \boldsymbol{\epsilon}_t, \sigma\right) \tag{70}$$

$$\mathbf{x}_t = \mathbf{h}_1\left(\mathbf{x}_{t-1}, \boldsymbol{\epsilon}_t, \sigma\right) \tag{71}$$

$$\boldsymbol{\epsilon}_{t+1} = \mathbf{u}_{t+1} \tag{72}$$

which is the Dynare and Dynare ++ notation with innovations entering nonlinearly. Accordingly, we can without loss of generality consider the SGU-notation.

We next illustrate how the Dynare-notation can be implemented with SGU-codes for the simple neoclassical model.The standard implementation reads:

$$f \equiv \begin{bmatrix} c_t + k_{t+1} - (1-\delta) k_t - a_t k_t^{\alpha} \\ c_t^{-\gamma} - \beta c_{t+1}^{-\gamma} \left(a_{t+1} \alpha k_{t+1}^{\alpha-1} + 1 - \delta \right) \\ \log a_{t+1} - \rho \log a_t \end{bmatrix}$$

where $\mathbf{x}_t \equiv \begin{bmatrix} k_t & a_t \end{bmatrix}$ and $\mathbf{y}_t \equiv [c_t]$. The equivalent Dynare-notation implementation is given by

$$f \equiv \begin{bmatrix} c_t + k_{t+1} - (1-\delta) k_t - a_t k_t^{\alpha} \\ c_t^{-\gamma} - \beta c_{t+1}^{-\gamma} \left(\exp\left\{ \rho \log a_t + \sigma \boldsymbol{\epsilon}_{t+1} \right\} \alpha k_{t+1}^{\alpha-1} + 1 - \delta \right) \\ \log a_t - \rho \log a_{t-1} - \sigma \boldsymbol{\epsilon}_t \\ \boldsymbol{\epsilon}_{t+1} \end{bmatrix}$$

where $\mathbf{x}_t \equiv \begin{bmatrix} k_t & a_{t-1} & \boldsymbol{\epsilon}_t \end{bmatrix}$ and $\mathbf{y}_t \equiv [c_t]$.

9 Existence of Skewness and Kurtosis

This section derives conditions for the existence of skewness and kurtosis in a linear system. We consider the system $\mathbf{x}_{t+1} = \mathbf{a} + \mathbf{A}\mathbf{x}_t + \mathbf{v}_{t+1}$ where $\mathbf{A}$ is stable and $\mathbf{v}_{t+1}$ are mean-zero innovations. Thus, the pruned state-space representation for DSGE models belong to this class. For notational convience, the system is re-express in deviation from its mean as $(\mathbf{I} - \mathbf{A}) E[\mathbf{x}] = \mathbf{a}$ and therefore

$$\mathbf{x}_{t+1} = (\mathbf{I} - \mathbf{A}) E[\mathbf{x}] + \mathbf{A}\mathbf{x}_t + \mathbf{v}_{t+1}$$
$$\Updownarrow$$
$$\mathbf{x}_{t+1} - E[\mathbf{x}] = \mathbf{A}(\mathbf{x}_t - E[\mathbf{x}]) + \mathbf{v}_{t+1}$$
$$\Updownarrow$$
$$\mathbf{z}_{t+1} = \mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1}$$

We then have

$$\mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1} = (\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1}) \otimes (\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1})$$

$$= \mathbf{A}\mathbf{z}_t \otimes (\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1}) + \mathbf{v}_{t+1} \otimes (\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1})$$

$$= \mathbf{A}\mathbf{z}_t \otimes \mathbf{A}\mathbf{z}_t + \mathbf{A}\mathbf{z}_t \otimes \mathbf{v}_{t+1} + \mathbf{v}_{t+1} \otimes \mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1}$$

$$\mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1} = (\mathbf{A}\mathbf{z}_t \otimes \mathbf{A}\mathbf{z}_t + \mathbf{A}\mathbf{z}_t \otimes \mathbf{v}_{t+1} + \mathbf{v}_{t+1} \otimes \mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1}) \otimes (\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1})$$

$$= \mathbf{A}\mathbf{z}_t \otimes \mathbf{A}\mathbf{z}_t \otimes (\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1}) + \mathbf{A}\mathbf{z}_t \otimes \mathbf{v}_{t+1} \otimes (\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1}) + \mathbf{v}_{t+1} \otimes \mathbf{A}\mathbf{z}_t \otimes (\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1}) + \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes (\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1})$$

$$\begin{aligned} &= \mathbf{A}\mathbf{z}_t \otimes \mathbf{A}\mathbf{z}_t \otimes \mathbf{A}\mathbf{z}_t + \mathbf{A}\mathbf{z}_t \otimes \mathbf{A}\mathbf{z}_t \otimes \mathbf{v}_{t+1} \\ &+ \mathbf{A}\mathbf{z}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{A}\mathbf{z}_t + \mathbf{A}\mathbf{z}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \\ &+ \mathbf{v}_{t+1} \otimes \mathbf{A}\mathbf{z}_t \otimes \mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1} \otimes \mathbf{A}\mathbf{z}_t \otimes \mathbf{v}_{t+1} \\ &+ \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \end{aligned}$$

Thus, to solve for $E[\mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1}]$ the innovations need to have a finite third moment. At second order, $\mathbf{v}_{t+1}$ is function of $\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}$, meaning that $\boldsymbol{\epsilon}_{t+1}$ must have finite sixth moment. At third order, $\mathbf{v}_{t+1}$ is function of $\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}$, meaning that $\boldsymbol{\epsilon}_{t+1}$ must have finite ninth moment.

$$\mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1}$$

$$= (\mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t + \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1}$$
$$+\mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t + \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1}$$
$$+\mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t + \mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1}$$
$$+\mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t + \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1}) \otimes (\mathbf{Az}_t + \mathbf{v}_{t+1})$$

$$= \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes (\mathbf{Az}_t + \mathbf{v}_{t+1}) + \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes (\mathbf{Az}_t + \mathbf{v}_{t+1})$$
$$+\mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes (\mathbf{Az}_t + \mathbf{v}_{t+1}) + \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes (\mathbf{Az}_t + \mathbf{v}_{t+1})$$
$$+\mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes (\mathbf{Az}_t + \mathbf{v}_{t+1}) + \mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes (\mathbf{Az}_t + \mathbf{v}_{t+1})$$
$$+\mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes (\mathbf{Az}_t + \mathbf{v}_{t+1}) + \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes (\mathbf{Az}_t + \mathbf{v}_{t+1})$$

$$= \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t + \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1}$$
$$+\mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t + \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1}$$
$$+\mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t + \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1}$$
$$+\mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t + \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1}$$
$$+\mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t + \mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1}$$
$$+\mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t + \mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1}$$
$$+\mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t + \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1}$$
$$+\mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t + \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1}$$

Thus, to solve for $E[\mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1}]$ the innovations need to have a finite fourth moment. At second order, $\mathbf{v}_{t+1}$ is function of $\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}$, meaning that $\boldsymbol{\epsilon}_{t+1}$ must have finite eight moment. At third order, $\mathbf{v}_{t+1}$ is function of $\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}$, meaning that $\boldsymbol{\epsilon}_{t+1}$ must have finite twelve moment.

10 Impulse response functions - the definition by Andreasen

This section derives closed-form solutions for the impulse response function in non-linear DSGE models. Note that this section uses the definition of an impulse response function suggested by Andreasen. This definition is

$$\begin{aligned} IRF_{\mathbf{var}}(l, \boldsymbol{\nu}, \mathbf{w}_t) &= E_t\left[\mathbf{var}_{t+l} | \boldsymbol{\epsilon}_{t+1} = \boldsymbol{\epsilon}_{t+1} + \boldsymbol{\nu}, \boldsymbol{\epsilon}_{t+2}, \boldsymbol{\epsilon}_{t+3}, ..., \boldsymbol{\epsilon}_{t+l}\right] \\ &\quad - E_t\left[\mathbf{var}_{t+l} | \boldsymbol{\epsilon}_{t+1} = \boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}_{t+2}, \boldsymbol{\epsilon}_{t+3}, ..., \boldsymbol{\epsilon}_{t+l}\right] \end{aligned}$$

To reduce the notational burden in the derivations below, we adopt the parsimonious notation

$$IRF_{\mathbf{var}}(l, \boldsymbol{\nu}, \mathbf{w}_t) = E_t\left[\widetilde{\mathbf{var}}_{t+l}\right] - E_t\left[\mathbf{var}_{t+l}\right]$$

in relation to the conditional expectation operators.

10.1 At first order

Recall that we have:

$$\mathbf{x}^f_{t+1} = \mathbf{h_x}\mathbf{x}^f_t + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$

and

$$\begin{aligned} \mathbf{x}^f_{t+2} &= \mathbf{h_x}\mathbf{x}^f_{t+1} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2} \\ &= \mathbf{h_x}\left(\mathbf{h_x}\mathbf{x}^f_t + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2} \\ &= \mathbf{h}^2_{\mathbf{x}}\mathbf{x}^f_t + \mathbf{h_x}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2} \end{aligned}$$

and

$$\begin{aligned} \mathbf{x}^f_{t+3} &= \mathbf{h_x}\mathbf{x}^f_{t+2} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+3} \\ &= \mathbf{h_x}\left(\mathbf{h}^2_{\mathbf{x}}\mathbf{x}^f_t + \mathbf{h_x}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2}\right) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+3} \end{aligned}$$

$= \mathbf{h}_{\mathbf{x}}^{3}\mathbf{x}_{t}^{f} + \mathbf{h}_{\mathbf{x}}^{2}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \mathbf{h}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+3}$

$= \mathbf{h}_{\mathbf{x}}^{3}\mathbf{x}_{t}^{f} + \sum_{j=1}^{3}\mathbf{h}_{\mathbf{x}}^{3-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$

In general

$\mathbf{x}_{t+l}^{f} = \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f} + \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$

With a shock of $\boldsymbol{\nu}$ in period $t+1$, we have

$\tilde{\mathbf{x}}_{t+l}^{f} = \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f} + \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right)$

where we define $\boldsymbol{\delta}_{t}$ such that:

$\boldsymbol{\delta}_{t+j} = \boldsymbol{\nu}$ for $j = 1$

$\boldsymbol{\delta}_{t+j} = \mathbf{0}$ for $j \neq 1$

So

$E_t\left[\tilde{\mathbf{x}}_{t+l}^{f} - \mathbf{x}_{t+l}^{f}\right] = E_t\left[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right) - \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]$

$= \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$

$= \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$

using the definition of $\boldsymbol{\delta}_{t+j}$.

$= \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$

because $\boldsymbol{\delta}_{t+1} = \boldsymbol{\nu}$

and

$E_t\left[\tilde{\mathbf{y}}_{t+l}^{f} - \mathbf{y}_{t+l}^{f}\right] = \mathbf{g}_{\mathbf{x}}E_t\left[\tilde{\mathbf{x}}_{t+l}^{f} - \mathbf{x}_{t+l}^{f}\right]$

10.2 At second order

We need to consider:

$\mathbf{x}_{t+1}^{s} = \mathbf{h}_{\mathbf{x}}\mathbf{x}_{t}^{s} + \frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}$

$\mathbf{x}_{t+2}^{s} = \mathbf{h}_{\mathbf{x}}\mathbf{x}_{t+1}^{s} + \frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}$

$= \mathbf{h}_{\mathbf{x}}\left(\mathbf{h}_{\mathbf{x}}\mathbf{x}_{t}^{s} + \frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right) + \frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}$

$= \mathbf{h}_{\mathbf{x}}^{2}\mathbf{x}_{t}^{s} + \mathbf{h}_{\mathbf{x}}\frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right) + \mathbf{h}_{\mathbf{x}}\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2} + \frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}$

$\mathbf{x}_{t+3}^{s} = \mathbf{h}_{\mathbf{x}}\mathbf{x}_{t+2}^{s} + \frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}$

$= \mathbf{h}_{\mathbf{x}}\left(\mathbf{h}_{\mathbf{x}}^{2}\mathbf{x}_{t}^{s} + \mathbf{h}_{\mathbf{x}}\frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right) + \mathbf{h}_{\mathbf{x}}\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2} + \frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)$

$+ \frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}$

$= \mathbf{h}_{\mathbf{x}}^{3}\mathbf{x}_{t}^{s} + \mathbf{h}_{\mathbf{x}}^{2}\frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right) + \mathbf{h}_{\mathbf{x}}^{2}\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2} + \mathbf{h}_{\mathbf{x}}\frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right) + \mathbf{h}_{\mathbf{x}}\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}$

$+ \frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}$

$$= \mathbf{h}_{\mathbf{x}}^3\mathbf{x}_t^s + \mathbf{h}_{\mathbf{x}}^2\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f\right)$$
$$+\mathbf{h}_{\mathbf{x}}^2\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$

$$= \mathbf{h}_{\mathbf{x}}^3\mathbf{x}_t^s + \sum_{j=0}^{2}\mathbf{h}_{\mathbf{x}}^{2-j}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f\right) + \left(\sum_{j=0}^{2}\mathbf{h}_{\mathbf{x}}^{2-j}\right)\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$

and in general

$$\mathbf{x}_{t+l}^s = \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^s + \sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f\right) + \left(\sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\right)\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$

for $l = 1, 2, 3, ...$

Thus, to compute $E_t\left[\tilde{\mathbf{x}}_{t+l}^s - \mathbf{x}_{t+l}^s\right]$, we need to find $E_t\left[\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f\right]$. Hence, consider:

$$\mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f = \left(\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f + \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f + \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$
$$= \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+ \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f + \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

and

$$\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f = \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right)$$
$$+ \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f + \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right) \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right)$$

$$= \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f \otimes \left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$+ \left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f$$
$$+ \left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right) \otimes \left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$

This means that:

$$E_t\left[\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f\right]$$

$$= E_t[\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f \otimes \left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$+ \left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f$$
$$+ \left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right) \otimes \left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$-\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f - \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

$$-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}]$$

$$=\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$

$$=\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$

because $\boldsymbol{\delta}_{t+1}=\boldsymbol{\nu}$

Thus $E_t\left[\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f-\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^f\right]=\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$
Or (using another index)
$E_t\left[\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f-\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\right]=\mathbf{h}_{\mathbf{x}}^{j}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{j-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{j-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{j}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{j-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{j-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$
for $j=1,2,3,...$

Thus, we have in general
$E_t\left[\tilde{\mathbf{x}}_{t+l}^s-\mathbf{x}_{t+l}^s\right]=E_t\left[\sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f\right)-\sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\right)\right]$

$$=\left[\sum_{j=1}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\frac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f-\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\right]\right]$$

the shock hits in period $t+1$, so $\left(\tilde{\mathbf{x}}_t^f\otimes\tilde{\mathbf{x}}_t^f\right)=\mathbf{x}_t^f\otimes\mathbf{x}_t^f$

$$=\sum_{j=1}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{h}_{\mathbf{x}}^{j}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{j-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{j-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{j}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{j-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{j-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right)$$

$$= \sum_{j=1}^{l-1} \mathbf{h}_{\mathbf{x}}^{l-1-j} \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\mathbf{h}_{\mathbf{x}}^{j} \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{j-1} \sigma \boldsymbol{\eta\nu} + \mathbf{h}_{\mathbf{x}}^{j-1} \sigma \boldsymbol{\eta\nu} \otimes \mathbf{h}_{\mathbf{x}}^{j} \mathbf{x}_t^f + \left(\mathbf{h}_{\mathbf{x}}^{j-1} \otimes \mathbf{h}_{\mathbf{x}}^{j-1} \right) (\sigma \boldsymbol{\eta\nu} \otimes \sigma \boldsymbol{\eta\nu}) \right)$$

If we restrict the focus and do the IRF's at the unconditional mean of $\mathbf{x}_t^f = \mathbf{0}$, then we get

$$E_t \left[\tilde{\mathbf{x}}_{t+l}^s - \mathbf{x}_{t+l}^s \right] = \sum_{j=1}^{l-1} \mathbf{h}_{\mathbf{x}}^{l-1-j} \frac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\left(\mathbf{h}_{\mathbf{x}}^{j-1} \otimes \mathbf{h}_{\mathbf{x}}^{j-1} \right) (\sigma \boldsymbol{\eta\nu} \otimes \sigma \boldsymbol{\eta\nu}) \right)$$

When implementing the IRF, it may be useful to have a recursive expression. Here, it is must convenient to use $E_t \left[\tilde{\mathbf{x}}_{t+l}^s - \mathbf{x}_{t+l}^s \right] = \sum_{j=1}^{l-1} \mathbf{h}_{\mathbf{x}}^{l-1-j} \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^f - \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \right]$

So

$E_t \left[\tilde{\mathbf{x}}_{t+1}^s - \mathbf{x}_{t+1}^s \right] = 0$

$$\begin{aligned} E_t \left[\tilde{\mathbf{x}}_{t+2}^s - \mathbf{x}_{t+2}^s \right] &= \sum_{j=1}^{1} \mathbf{h}_{\mathbf{x}}^{1-j} \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^f - \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \right] \\ &= \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+1}^f \otimes \tilde{\mathbf{x}}_{t+1}^f - \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \right] \end{aligned}$$

$$\begin{aligned} E_t \left[\tilde{\mathbf{x}}_{t+3}^s - \mathbf{x}_{t+3}^s \right] &= \sum_{j=1}^{2} \mathbf{h}_{\mathbf{x}}^{2-j} \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^f - \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \right] \\ &= \mathbf{h}_{\mathbf{x}} \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+1}^f \otimes \tilde{\mathbf{x}}_{t+1}^f - \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \right] \\ &+ \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+2}^f \otimes \tilde{\mathbf{x}}_{t+2}^f - \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \right] \\ &= \mathbf{h}_{\mathbf{x}} E_t \left[\tilde{\mathbf{x}}_{t+2}^s - \mathbf{x}_{t+2}^s \right] + \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+2}^f \otimes \tilde{\mathbf{x}}_{t+2}^f - \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \right] \end{aligned}$$

So in general

$E_t \left[\tilde{\mathbf{x}}_{t+k}^s - \mathbf{x}_{t+k}^s \right] = \mathbf{h}_{\mathbf{x}} E_t \left[\tilde{\mathbf{x}}_{t+k-1}^s - \mathbf{x}_{t+k-1}^s \right] + \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+k-1}^f \otimes \tilde{\mathbf{x}}_{t+k-1}^f - \mathbf{x}_{t+k-1}^f \otimes \mathbf{x}_{t+k-1}^f \right]$

For the total state variable:

$E_t \left[\tilde{\mathbf{x}}_{t+l} - \mathbf{x}_{t+l} \right] = E_t \left[\tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f \right] + E_t \left[\tilde{\mathbf{x}}_{t+l}^s - \mathbf{x}_{t+l}^s \right]$

For the control variables:

$\mathbf{y}_{t+l}^s = \mathbf{g}_{\mathbf{x}} \left(\mathbf{x}_{t+l}^f + \mathbf{x}_{t+l}^s \right) + \tfrac{1}{2} \mathbf{G}_{\mathbf{xx}} \left(\mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f \right) + \tfrac{1}{2} \mathbf{g}_{\sigma\sigma} \sigma^2$

$\tilde{\mathbf{y}}_{t+l}^s = \mathbf{g}_{\mathbf{x}} \left(\tilde{\mathbf{x}}_{t+l}^f + \tilde{\mathbf{x}}_{t+l}^s \right) + \tfrac{1}{2} \mathbf{G}_{\mathbf{xx}} \left(\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f \right) + \tfrac{1}{2} \mathbf{g}_{\sigma\sigma} \sigma^2$

$$E_t \left[\tilde{\mathbf{y}}_{t+l}^s - \mathbf{y}_{t+l}^s \right] = \mathbf{g}_{\mathbf{x}} \left(E_t \left[\tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f \right] + E_t \left[\tilde{\mathbf{x}}_{t+l}^s - \mathbf{x}_{t+l}^s \right] \right) + \tfrac{1}{2} \mathbf{G}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f \right]$$

10.3 At third order

At third order, we additionally need to consider:

$$\mathbf{x}_{t+1}^{rd} = \mathbf{h}_{\mathbf{x}} \mathbf{x}_t^{rd} + \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s \right) + \tfrac{1}{6} \mathbf{H}_{\mathbf{xxx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f \right) + \tfrac{3}{6} \mathbf{h}_{\sigma\sigma\mathbf{x}} \sigma^2 \mathbf{x}_t^f + \tfrac{1}{6} \mathbf{h}_{\sigma\sigma\sigma} \sigma^3$$

and

$$\mathbf{x}_{t+2}^{rd} = \mathbf{h}_{\mathbf{x}} \mathbf{x}_{t+1}^{rd} + \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s \right) + \tfrac{1}{6} \mathbf{H}_{\mathbf{xxx}} \left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \right) + \tfrac{3}{6} \mathbf{h}_{\sigma\sigma\mathbf{x}} \sigma^2 \mathbf{x}_{t+1}^f + \tfrac{1}{6} \mathbf{h}_{\sigma\sigma\sigma} \sigma^3$$

$$= \mathbf{h_x}\left(\mathbf{h_x}\mathbf{x}_t^{rd} + \mathbf{H_{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\right)$$
$$+\mathbf{H_{xx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+1}^f + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

$$= \mathbf{h}_\mathbf{x}^2\mathbf{x}_t^{rd} + \mathbf{h_x}\mathbf{H_{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \mathbf{h_x}\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \mathbf{h_x}\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f + \mathbf{h_x}\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$
$$+\mathbf{H_{xx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+1}^f + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

and

$$\mathbf{x}_{t+3}^{rd} = \mathbf{h_x}\mathbf{x}_{t+2}^{rd} + \mathbf{H_{xx}}\left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^s\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+2}^f + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

$$= \mathbf{h_x}[\mathbf{h}_\mathbf{x}^2\mathbf{x}_t^{rd} + \mathbf{h_x}\mathbf{H_{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \mathbf{h_x}\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \mathbf{h_x}\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f + \mathbf{h_x}\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$
$$+\mathbf{H_{xx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+1}^f + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3]$$
$$+\mathbf{H_{xx}}\left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^s\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+2}^f + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

$$= \mathbf{h}_\mathbf{x}^3\mathbf{x}_t^{rd} + \mathbf{h}_\mathbf{x}^2\mathbf{H_{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \mathbf{h}_\mathbf{x}^2\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \mathbf{h}_\mathbf{x}^2\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f + \mathbf{h}_\mathbf{x}^2\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$
$$+\mathbf{h_x}\mathbf{H_{xx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s\right) + \mathbf{h_x}\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \mathbf{h_x}\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+1}^f + \mathbf{h_x}\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$
$$+\mathbf{H_{xx}}\left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^s\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+2}^f + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

$$= \mathbf{h}_\mathbf{x}^3\mathbf{x}_t^{rd} + \sum_{j=0}^{2}\mathbf{h}_\mathbf{x}^{2-j}\mathbf{H_{xx}}\left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^s\right)$$
$$+ \sum_{j=0}^{2}\mathbf{h}_\mathbf{x}^{2-j}\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f\right)$$
$$+ \sum_{j=0}^{2}\mathbf{h}_\mathbf{x}^{2-j}\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+j}^f$$
$$+ \sum_{j=0}^{2}\mathbf{h}_\mathbf{x}^{2-j}\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

$$= \mathbf{h}_\mathbf{x}^3\mathbf{x}_t^{rd} + \sum_{j=0}^{2}\mathbf{h}_\mathbf{x}^{2-j}\left[\mathbf{H_{xx}}\left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^s\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+j}^f\right]$$
$$+ \left(\sum_{j=0}^{2}\mathbf{h}_\mathbf{x}^{2-j}\right)\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

In general

$$\mathbf{x}_{t+l}^{rd} = \mathbf{h}_\mathbf{x}^{l}\mathbf{x}_t^{rd} + \sum_{j=0}^{l-1}\mathbf{h}_\mathbf{x}^{l-1-j}\left[\mathbf{H_{xx}}\left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^s\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+j}^f\right]$$
$$+ \left(\sum_{j=0}^{l-1}\mathbf{h}_\mathbf{x}^{l-1-j}\right)\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

Thus

$$E_t\left[\tilde{\mathbf{x}}_{t+l}^{rd} - \mathbf{x}_{t+l}^{rd}\right] = E_t[\sum_{j=0}^{l-1}\mathbf{h}_\mathbf{x}^{l-1-j}\left[\mathbf{H_{xx}}\left(\tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^s\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\tilde{\mathbf{x}}_{t+j}^f\right]$$
$$-\{\sum_{j=0}^{l-1}\mathbf{h}_\mathbf{x}^{l-1-j}\left[\mathbf{H_{xx}}\left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^s\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+j}^f\right]\}]$$

$$= \sum_{j=1}^{l-1} \mathbf{h}_{\mathbf{x}}^{l-1-j} \left(\mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^s - \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^s \right] + \tfrac{3}{6} \mathbf{h}_{\sigma\sigma\mathbf{x}} \sigma^2 E_t \left[\tilde{\mathbf{x}}_{t+j}^f - \mathbf{x}_{t+j}^f \right] \right)$$
$$+ \sum_{j=1}^{l-1} \mathbf{h}_{\mathbf{x}}^{l-1-j} \tfrac{1}{6} \mathbf{H}_{\mathbf{xxx}} E_t \left[\tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^f - \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \right]$$

as the shock hits the economy in period $t+1$

A recursive version:
$E_t \left[\tilde{\mathbf{x}}_{t+1}^{rd} - \mathbf{x}_{t+1}^{rd} \right] = 0$

$$E_t \left[\tilde{\mathbf{x}}_{t+2}^{rd} - \mathbf{x}_{t+2}^{rd} \right] = \sum_{j=1}^{1} \mathbf{h}_{\mathbf{x}}^{1-j} \left(\mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^s - \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^s \right] + \tfrac{3}{6} \mathbf{h}_{\sigma\sigma\mathbf{x}} \sigma^2 E_t \left[\tilde{\mathbf{x}}_{t+j}^f - \mathbf{x}_{t+j}^f \right] \right)$$
$$+ \sum_{j=1}^{1} \mathbf{h}_{\mathbf{x}}^{1-j} \tfrac{1}{6} \mathbf{H}_{\mathbf{xxx}} E_t \left[\tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^f - \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \right]$$
$$= \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+1}^f \otimes \tilde{\mathbf{x}}_{t+1}^s - \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s \right] + \tfrac{3}{6} \mathbf{h}_{\sigma\sigma\mathbf{x}} \sigma^2 E_t \left[\tilde{\mathbf{x}}_{t+1}^f - \mathbf{x}_{t+1}^f \right]$$
$$+ \tfrac{1}{6} \mathbf{H}_{\mathbf{xxx}} E_t \left[\tilde{\mathbf{x}}_{t+1}^f \otimes \tilde{\mathbf{x}}_{t+1}^f \otimes \tilde{\mathbf{x}}_{t+1}^f - \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \right]$$

$$E_t \left[\tilde{\mathbf{x}}_{t+3}^{rd} - \mathbf{x}_{t+3}^{rd} \right] = \sum_{j=1}^{2} \mathbf{h}_{\mathbf{x}}^{2-j} \left(\mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^s - \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^s \right] + \tfrac{3}{6} \mathbf{h}_{\sigma\sigma\mathbf{x}} \sigma^2 E_t \left[\tilde{\mathbf{x}}_{t+j}^f - \mathbf{x}_{t+j}^f \right] \right)$$
$$+ \sum_{j=1}^{2} \mathbf{h}_{\mathbf{x}}^{2-j} \tfrac{1}{6} \mathbf{H}_{\mathbf{xxx}} E_t \left[\tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^f - \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \right]$$
$$= \mathbf{h}_{\mathbf{x}} \left(\mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+1}^f \otimes \tilde{\mathbf{x}}_{t+1}^s - \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s \right] + \tfrac{3}{6} \mathbf{h}_{\sigma\sigma\mathbf{x}} \sigma^2 E_t \left[\tilde{\mathbf{x}}_{t+1}^f - \mathbf{x}_{t+1}^f \right] \right)$$
$$+ \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+2}^f \otimes \tilde{\mathbf{x}}_{t+2}^s - \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^s \right] + \tfrac{3}{6} \mathbf{h}_{\sigma\sigma\mathbf{x}} \sigma^2 E_t \left[\tilde{\mathbf{x}}_{t+2}^f - \mathbf{x}_{t+2}^f \right]$$
$$+ \mathbf{h}_{\mathbf{x}} \tfrac{1}{6} \mathbf{H}_{\mathbf{xxx}} E_t \left[\tilde{\mathbf{x}}_{t+1}^f \otimes \tilde{\mathbf{x}}_{t+1}^f \otimes \tilde{\mathbf{x}}_{t+1}^f - \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \right]$$
$$+ \tfrac{1}{6} \mathbf{H}_{\mathbf{xxx}} E_t \left[\tilde{\mathbf{x}}_{t+2}^f \otimes \tilde{\mathbf{x}}_{t+2}^f \otimes \tilde{\mathbf{x}}_{t+2}^f - \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \right]$$

$$= \mathbf{h}_{\mathbf{x}} E_t \left[\tilde{\mathbf{x}}_{t+2}^{rd} - \mathbf{x}_{t+2}^{rd} \right]$$
$$+ \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+2}^f \otimes \tilde{\mathbf{x}}_{t+2}^s - \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^s \right] + \tfrac{3}{6} \mathbf{h}_{\sigma\sigma\mathbf{x}} \sigma^2 E_t \left[\tilde{\mathbf{x}}_{t+2}^f - \mathbf{x}_{t+2}^f \right]$$
$$+ \tfrac{1}{6} \mathbf{H}_{\mathbf{xxx}} E_t \left[\tilde{\mathbf{x}}_{t+2}^f \otimes \tilde{\mathbf{x}}_{t+2}^f \otimes \tilde{\mathbf{x}}_{t+2}^f - \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \right]$$

So in general

$$E_t \left[\tilde{\mathbf{x}}_{t+k}^{rd} - \mathbf{x}_{t+k}^{rd} \right] = \mathbf{h}_{\mathbf{x}} E_t \left[\tilde{\mathbf{x}}_{t+k-1}^{rd} - \mathbf{x}_{t+k-1}^{rd} \right] + \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+k-1}^f \otimes \tilde{\mathbf{x}}_{t+k-1}^s - \mathbf{x}_{t+k-1}^f \otimes \mathbf{x}_{t+k-1}^s \right]$$
$$+ \tfrac{3}{6} \mathbf{h}_{\sigma\sigma\mathbf{x}} \sigma^2 E_t \left[\tilde{\mathbf{x}}_{t+k-1}^f - \mathbf{x}_{t+k-1}^f \right]$$
$$+ \tfrac{1}{6} \mathbf{H}_{\mathbf{xxx}} E_t \left[\tilde{\mathbf{x}}_{t+k-1}^f \otimes \tilde{\mathbf{x}}_{t+k-1}^f \otimes \tilde{\mathbf{x}}_{t+k-1}^f - \mathbf{x}_{t+k-1}^f \otimes \mathbf{x}_{t+k-1}^f \otimes \mathbf{x}_{t+k-1}^f \right]$$

Thus, we know $E_t \left[\tilde{\mathbf{x}}_{t+j}^f - \mathbf{x}_{t+j}^f \right]$. So we only need to compute $E_t \left[\tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^f - \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \right]$ and $E_t \left[\tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^s - \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^s \right]$. This is done in the next two subsections.

10.3.1 For $\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)$

Consider:

$$\mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f = \left(\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}\right)$$

$$= (\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j})$$

$$\otimes \left(\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}\right)$$

$$= (\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}) \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f$$

$$+(\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}) \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}$$

$$= \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f$$

$$+ \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f$$

$$+ \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}$$

$$+ \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}$$

And we therefore have

$$\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f = \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f$$

$$+ \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right) \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f$$

$$+ \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right) + \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right) \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right)$$

$$+ \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right)$$

$$+ \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right) \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right) \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right)$$

$$= \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f$$

$$+ \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f$$

$$+ \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1}\right) \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f$$

$$+ \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1}\right)$$

$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right) \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$+\left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$+\left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right) \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right) \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$

This means that:
$E_t\left[\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f\right]$

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right) \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right) \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$+\left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$+\left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right) \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right) \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)]$$

$$-E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right) \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right) \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$+\left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$

$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)]$$

$$-E_t[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right)$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+\left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)]$$

$$-E_t[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right)$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right)$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$

$$
\begin{aligned}
&+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\\
&+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)]
\end{aligned}
$$

$$
\begin{aligned}
&-E_t[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]
\end{aligned}
$$

$$
\begin{aligned}
&=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\\
&+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)]
\end{aligned}
$$

$$
\begin{aligned}
&-E_t[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}
\end{aligned}
$$

$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+0+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+\mathbf{0}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+0+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+\left(0+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}]$$

$-E_t[0$
$+\mathbf{0}$
$+0$
$+0]$
Terms cannel out

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+0$$
$$+\mathbf{0}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+\mathbf{0}$$
$$+\mathbf{0}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+0$$
$$+\mathbf{0}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+0$$

$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{0}$$
$$+\mathbf{0}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}]$$
using the zero-mean property of $\boldsymbol{\epsilon}_{t+j}$

$$\begin{aligned}
&= \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+E_t\left[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]\\
&+E_t\left[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right]\\
&+E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
&+E_t\left[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]\\
&+E_t\left[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right]\\
&+E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}
\end{aligned}$$
using that $\boldsymbol{\delta}_{t+1}=\boldsymbol{\nu}$

We then note that:

$$\begin{aligned}
&\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right)\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right)
\end{aligned}$$

$$= \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \left(\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} + \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \right)$$
$$+ \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \left(\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \right)$$
$$+ \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \otimes \left(\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \right)$$
$$+ \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu}$$

$$= \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \left(\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \right) \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu}$$
$$+ \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \left(\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \right)$$
$$+ \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \otimes \left(\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \right) \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f$$
$$+ \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu}$$

Thus
$E_t \left[\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f \right]$

$$= \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \left(\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \right) \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu}$$
$$+ \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f$$
$$+ \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \otimes \left(\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \right) \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f$$
$$+ \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu}$$
$$+ E_t \left[\sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta\epsilon}_{t+j} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \otimes \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta\epsilon}_{t+j} \right]$$
$$+ E_t \left[\sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta\epsilon}_{t+j} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \right]$$
$$+ E_t \left[\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \otimes \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta\epsilon}_{t+j} \right]$$
$$+ \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu}$$

The final three terms can be computed as follows:
$E_t \left[\sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta\epsilon}_{t+j} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \otimes \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta\epsilon}_{t+j} \right]$

$$= E_t[\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\epsilon}_{t+1} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \otimes \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta\epsilon}_{t+j}$$
$$+ \mathbf{h}_\mathbf{x}^{l-2} \sigma \boldsymbol{\eta\epsilon}_{t+2} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \otimes \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta\epsilon}_{t+j}$$
$$+ \ldots$$
$$+ \sigma \boldsymbol{\eta\epsilon}_{t+l} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \otimes \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta\epsilon}_{t+j}]$$

$$= E_t[\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\epsilon}_{t+1} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\epsilon}_{t+1}$$
$$+ \mathbf{h}_\mathbf{x}^{l-2} \sigma \boldsymbol{\eta\epsilon}_{t+2} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \otimes \mathbf{h}_\mathbf{x}^{l-2} \sigma \boldsymbol{\eta\epsilon}_{t+2}$$
$$+ \ldots$$
$$+ \sigma \boldsymbol{\eta\epsilon}_{t+l} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta\nu} \otimes \sigma \boldsymbol{\eta\epsilon}_{t+l}]$$

because the innovations are independent across time

$$= \sum_{j=1}^{l} E_t[\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$= \sum_{j=1}^{l} \boldsymbol{\Omega}_j$$

Let $\boldsymbol{\Omega}_j \equiv E_t\left[\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$. So

$$\boldsymbol{\Omega}_j \equiv E_t\left[\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_2,:)\sigma\boldsymbol{\eta}\boldsymbol{\nu} \times \left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right]$$

$$= E_t\left\{\mathbf{h}_{\mathbf{x}}^{l-j}\left(\gamma_{3,:}\right)\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \times \left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_2,:)\sigma\boldsymbol{\eta}\boldsymbol{\nu} \times \left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}$$

$$= E_t\left\{\mathbf{h}_{\mathbf{x}}^{l-j}\left(\gamma_{3,:}\right)\sigma\sum_{\phi_2=1}^{n_e}\boldsymbol{\eta}(:,\phi_2)\boldsymbol{\epsilon}_{t+1}(\phi_2,1) \times \left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_2,:)\sigma\boldsymbol{\eta}\boldsymbol{\nu} \times \left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\sigma\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}(:,\phi_1)\boldsymbol{\epsilon}_{t+1}(\phi_1,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}$$

$$= E_t\left\{\mathbf{h}_{\mathbf{x}}^{l-j}\left(\gamma_{3,:}\right)\sigma\sum_{\phi_2=1}^{n_e}\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}(:,\phi_2)\boldsymbol{\epsilon}_{t+1}(\phi_2,1) \times \left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_2,:)\sigma\boldsymbol{\eta}\boldsymbol{\nu} \times \left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\sigma\boldsymbol{\eta}(:,\phi_1)\boldsymbol{\epsilon}_{t+1}(\phi_1,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}$$

$$= \left\{\mathbf{h}_{\mathbf{x}}^{l-j}\left(\gamma_{3,:}\right)\sigma\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}(:,\phi_1) \times \left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_2,:)\sigma\boldsymbol{\eta}\boldsymbol{\nu} \times \left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\sigma\boldsymbol{\eta}(:,\phi_1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}$$

because the innovations are indepdendent. This expression is directly implementable.

And

$$E_t\left[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right]$$

$$= \sum_{j=1}^{l} E_t[\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}]$$

$$= \sum_{j=1}^{l} E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-j} \otimes \mathbf{h}_{\mathbf{x}}^{l-j}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}]$$

$$= \sum_{j=1}^{l} \left(\mathbf{h}_{\mathbf{x}}^{l-j} \otimes \mathbf{h}_{\mathbf{x}}^{l-j}\right) E_t\left[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right] \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$

$$= \sum_{j=1}^{l} \left(\mathbf{h}_{\mathbf{x}}^{l-j} \otimes \mathbf{h}_{\mathbf{x}}^{l-j}\right) \boldsymbol{\Lambda} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$

where $\boldsymbol{\Lambda} \equiv E_t\left[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$. To compute $\boldsymbol{\Lambda}$ we note that

$$\boldsymbol{\Lambda} = E_t\left[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$$

$$= E_t\left\{\left\{\sigma\boldsymbol{\eta}(\gamma_2,:)\boldsymbol{\epsilon}_{t+1}\sigma\boldsymbol{\eta}(\gamma_1,:)\boldsymbol{\epsilon}_{t+1}\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}$$

$$= E_t\left\{\left\{\sigma\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}(\gamma_2,\phi_1)\boldsymbol{\epsilon}_{t+1}(\phi_1,1)\sum_{\phi_2=1}^{n_e}\sigma\boldsymbol{\eta}(\gamma_1,\phi_2)\boldsymbol{\epsilon}_{t+1}(\phi_2,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}$$

$$= E_t \left\{ \left\{ \sigma \sum_{\phi_1=1}^{n_e} \sum_{\phi_2=1}^{n_e} \boldsymbol{\eta} (\gamma_2, \phi_1) \boldsymbol{\epsilon}_{t+1} (\phi_1, 1) \sigma \boldsymbol{\eta} (\gamma_1, \phi_2) \boldsymbol{\epsilon}_{t+1} (\phi_2, 1) \right\}_{\gamma_1=1}^{n_x} \right\}_{\gamma_2=1}^{n_x}$$

$$= \left\{ \left\{ \sigma \sum_{\phi_1=1}^{n_e} \boldsymbol{\eta} (\gamma_2, \phi_1) \sigma \boldsymbol{\eta} (\gamma_1, \phi_1) \right\}_{\gamma_1=1}^{n_x} \right\}_{\gamma_2=1}^{n_x}$$

$$= \left\{ \left\{ \sigma^2 \boldsymbol{\eta} (\gamma_2, :) \boldsymbol{\eta} (\gamma_1, :)' \right\}_{\gamma_1=1}^{n_x} \right\}_{\gamma_2=1}^{n_x}$$

And finally

$$E_t \left[\mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \right]$$

$$= \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes E_t [\mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}]$$

$$= \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-j} \otimes \mathbf{h}_{\mathbf{x}}^{l-j} \right) E_t \left[\sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \right]$$

$$= \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-j} \otimes \mathbf{h}_{\mathbf{x}}^{l-j} \right) \boldsymbol{\Lambda}$$

10.3.2 For $\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s \right)$

Recall from above that

$$\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s = \left(\mathbf{h}_{\mathbf{x}} \mathbf{x}_t^f + \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \right) \otimes \left(\mathbf{h}_{\mathbf{x}} \mathbf{x}_t^s + \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \right) + \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 \right)$$

$$= (\mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_{\mathbf{x}}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s \right) + \left(\mathbf{h}_{\mathbf{x}} \otimes \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \right) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f \right) + \left(\mathbf{h}_{\mathbf{x}} \otimes \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 \right) \mathbf{x}_t^f$$
$$+ (\sigma \boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}) (\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s) + \left(\sigma \boldsymbol{\eta} \otimes \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f \right) + \left(\sigma \boldsymbol{\eta} \otimes \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 \right) \boldsymbol{\epsilon}_{t+1}$$

Therefore:

$$\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^s = (\mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_{\mathbf{x}}) \left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s \right) + \left(\mathbf{h}_{\mathbf{x}} \otimes \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \right) \left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \right) + \left(\mathbf{h}_{\mathbf{x}} \otimes \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 \right) \mathbf{x}_{t+1}^f$$
$$+ (\sigma \boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}) \left(\boldsymbol{\epsilon}_{t+2} \otimes \mathbf{x}_{t+1}^s \right) + \left(\sigma \boldsymbol{\eta} \otimes \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \right) \left(\boldsymbol{\epsilon}_{t+2} \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \right) + \left(\sigma \boldsymbol{\eta} \otimes \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 \right) \boldsymbol{\epsilon}_{t+2}$$

$$= (\mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_{\mathbf{x}}) [(\mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_{\mathbf{x}}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s \right) + \left(\mathbf{h}_{\mathbf{x}} \otimes \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \right) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f \right) + \left(\mathbf{h}_{\mathbf{x}} \otimes \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 \right) \mathbf{x}_t^f$$
$$+ (\sigma \boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}) (\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s) + \left(\sigma \boldsymbol{\eta} \otimes \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f \right) + \left(\sigma \boldsymbol{\eta} \otimes \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 \right) \boldsymbol{\epsilon}_{t+1}]$$

$$+ \left(\mathbf{h}_{\mathbf{x}} \otimes \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \right) \left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \right) + \left(\mathbf{h}_{\mathbf{x}} \otimes \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 \right) \mathbf{x}_{t+1}^f$$
$$+ (\sigma \boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}) \left(\boldsymbol{\epsilon}_{t+2} \otimes \mathbf{x}_{t+1}^s \right) + \left(\sigma \boldsymbol{\eta} \otimes \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \right) \left(\boldsymbol{\epsilon}_{t+2} \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \right) + \left(\sigma \boldsymbol{\eta} \otimes \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 \right) \boldsymbol{\epsilon}_{t+2}$$

$$= (\mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_{\mathbf{x}})^2 \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s \right) + (\mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_{\mathbf{x}}) \left(\mathbf{h}_{\mathbf{x}} \otimes \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \right) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f \right) + (\mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_{\mathbf{x}}) \left(\mathbf{h}_{\mathbf{x}} \otimes \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 \right) \mathbf{x}_t^f$$
$$+ (\mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_{\mathbf{x}}) (\sigma \boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}) (\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s) + (\mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_{\mathbf{x}}) \left(\sigma \boldsymbol{\eta} \otimes \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f \right) + (\mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_{\mathbf{x}}) \left(\sigma \boldsymbol{\eta} \otimes \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 \right) \boldsymbol{\epsilon}_{t+1}$$

$$+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right)+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}^f_{t+1}$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}^s_{t+1}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+2}$$

and

$$\mathbf{x}^f_{t+3}\otimes\mathbf{x}^s_{t+3}=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{x}^f_{t+2}\otimes\mathbf{x}^s_{t+2}\right)+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right)+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}^f_{t+2}$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}^s_{t+2}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+3}$$

$$=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)[\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^2\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}^f_t\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}^f_t$$
$$+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}^s_t\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}$$
$$+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right)+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}^f_{t+1}$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}^s_{t+1}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+2}]$$

$$+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right)+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}^f_{t+2}$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}^s_{t+2}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+3}$$

$$=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^3\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^2\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}^f_t\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^2\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}^f_t$$
$$+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^2\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}^s_t\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^2\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^2\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}$$
$$+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}^f_{t+1}$$
$$+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}^s_{t+1}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right)$$
$$+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+2}]$$

$$+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right)+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}^f_{t+2}$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}^s_{t+2}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+3}$$

$$=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^3\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)+\sum_{i=0}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)$$
$$+\sum_{i=0}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}^f_{t+i}$$
$$+\sum_{i=0}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1+i}$$
$$+\sum_{i=0}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^s_{t+i}\right)$$
$$+\sum_{i=0}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)$$

And in general

$$\mathbf{x}^f_{t+l}\otimes\mathbf{x}^s_{t+l}=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^l\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)$$
$$+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}^f_{t+i}$$

$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1+i}$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^s_{t+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)$$

We therefore have

$$\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^s_{t+l}=(\mathbf{h_x}\otimes\mathbf{h_x})^l\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\tilde{\mathbf{x}}^f_{t+i}$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\boldsymbol{\epsilon}_{t+1+i}+\boldsymbol{\delta}_{t+1+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left((\boldsymbol{\epsilon}_{t+1+i}+\boldsymbol{\delta}_{t+1+i})\otimes\tilde{\mathbf{x}}^s_{t+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left((\boldsymbol{\epsilon}_{t+1+i}+\boldsymbol{\delta}_{t+1+i})\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\right)$$

Thus

$$E_t\left[\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^s_{t+l}-\mathbf{x}^f_{t+l}\otimes\mathbf{x}^s_{t+l}\right]$$

$$=E_t[(\mathbf{h_x}\otimes\mathbf{h_x})^l\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\tilde{\mathbf{x}}^f_{t+i}$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\boldsymbol{\epsilon}_{t+1+i}+\boldsymbol{\delta}_{t+1+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left((\boldsymbol{\epsilon}_{t+1+i}+\boldsymbol{\delta}_{t+1+i})\otimes\tilde{\mathbf{x}}^s_{t+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left((\boldsymbol{\epsilon}_{t+1+i}+\boldsymbol{\delta}_{t+1+i})\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\right)$$

$$-\{(\mathbf{h_x}\otimes\mathbf{h_x})^l\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}^f_{t+i}$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1+i}$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^s_{t+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)\}$$
$$]$$

$$=E_t[\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\right)$$

$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\delta}_{t+1+i}$
$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left((\boldsymbol{\epsilon}_{t+1+i}+\boldsymbol{\delta}_{t+1+i})\otimes\tilde{\mathbf{x}}^s_{t+i}\right)$
$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{H_{xx}}\right)\left((\boldsymbol{\epsilon}_{t+1+i}+\boldsymbol{\delta}_{t+1+i})\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\right)$

$-\{\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^s_{t+i}\right)$
$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)\}$
]

$=E_t[\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\frac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)$
$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\right)$
$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\delta}_{t+1+i}$
$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\delta}_{t+1+i}\otimes\tilde{\mathbf{x}}^s_{t+i}\right)$
$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\delta}_{t+1+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\right)$
]
because $\mathbf{x}^s_{t+i}$ is a function of $\mathbf{x}^f_{t+i}$ which is a function of $\boldsymbol{\epsilon}_{t+i}$. The zero-mean iid innovations therefore implies that, $E_t\left[\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^s_{t+i}\right)\right]=\mathbf{0}$ and $E_t\left[\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\tilde{\mathbf{x}}^s_{t+i}\right)\right]=\mathbf{0}$
The same argument implies that $E_t\left[\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)\right]=\mathbf{0}$
and $E_t\left[\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\right)\right]=\mathbf{0}$

$=\sum_{i=1}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\frac{1}{2}\mathbf{H_{xx}}\right)E_t\left[\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right]$
$+\sum_{i=1}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)E_t\left[\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\right]$
$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\nu}$
$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\nu}\otimes E_t\left[\tilde{\mathbf{x}}^s_t\right]\right)$
$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\nu}\otimes E_t\left[\tilde{\mathbf{x}}^f_t\otimes\tilde{\mathbf{x}}^f_t\right]\right)$
using $\boldsymbol{\delta}_{t+1}=\boldsymbol{\nu}$ for else $\boldsymbol{\delta}_{t+1+i}=0$
and therefore the index for l starts at 1

$=\sum_{i=1}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\frac{1}{2}\mathbf{H_{xx}}\right)E_t\left[\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right]$
$+\sum_{i=1}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)E_t\left[\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\right]$
$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\nu}$
$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\nu}\otimes E_t\left[\tilde{\mathbf{x}}^s_t\right]\right)$
$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\nu}\otimes E_t\left[\tilde{\mathbf{x}}^f_t\otimes\tilde{\mathbf{x}}^f_t\right]\right)$

$$
\begin{aligned}
&= \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right] \\
&+ \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i}\right] \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \boldsymbol{\nu} \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right) \left(\boldsymbol{\nu} \otimes \mathbf{x}^s_t\right) \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\boldsymbol{\nu} \otimes \mathbf{x}^f_t \otimes \mathbf{x}^f_t\right)
\end{aligned}
$$

because the shock hits in period $t+1$, so $E_t\left[\tilde{\mathbf{x}}^s_t\right] = \mathbf{x}^s_t$ and $E_t\left[\tilde{\mathbf{x}}^f_t \otimes \tilde{\mathbf{x}}^f_t\right] = \mathbf{x}^f_t \otimes \mathbf{x}^f_t$

$$
\begin{aligned}
&= \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right] \\
&+ \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i}\right] \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \mathbf{h_x}\mathbf{x}^s_t\right) \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right] \\
&+ \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i}\right] \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \sigma\boldsymbol{\eta\nu} \otimes \mathbf{h_x}\mathbf{x}^s_t + \sigma\boldsymbol{\eta\nu} \otimes \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right] \\
&+ \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i}\right] \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x}\mathbf{x}^s_t + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)
\end{aligned}
$$

A recursive version for the sum reads:

$$
\begin{aligned}
X_l &= \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right] \\
&+ \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i}\right] \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x}\mathbf{x}^s_t + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)
\end{aligned}
$$

So

$$
X_1 = \sigma\boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x}\mathbf{x}^s_t + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)
$$

$$
\begin{aligned}
X_2 &= \sum_{i=1}^{1} (\mathbf{h_x} \otimes \mathbf{h_x})^{1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right] \\
&+ \sum_{i=1}^{1} (\mathbf{h_x} \otimes \mathbf{h_x})^{1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i}\right] \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\sigma\boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x}\mathbf{x}^s_t + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) E_t \left[\tilde{\mathbf{x}}^f_{t+1} \otimes \tilde{\mathbf{x}}^f_{t+1} \otimes \tilde{\mathbf{x}}^f_{t+1} - \mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1}\right] \\
&+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) E_t \left[\tilde{\mathbf{x}}^f_{t+1} - \mathbf{x}^f_{t+1}\right]
\end{aligned}
$$

$$+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta\nu}\otimes\left(\mathbf{h_x}\mathbf{x}_t^s+\tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)$$

$$=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)X_1+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)E_t\left[\tilde{\mathbf{x}}_{t+1}^f\otimes\tilde{\mathbf{x}}_{t+1}^f\otimes\tilde{\mathbf{x}}_{t+1}^f-\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right]$$
$$+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)E_t\left[\tilde{\mathbf{x}}_{t+1}^f-\mathbf{x}_{t+1}^f\right]$$

$$X_3=\sum_{i=1}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)E_t\left[\tilde{\mathbf{x}}_{t+i}^f\otimes\tilde{\mathbf{x}}_{t+i}^f\otimes\tilde{\mathbf{x}}_{t+i}^f-\mathbf{x}_{t+i}^f\otimes\mathbf{x}_{t+i}^f\otimes\mathbf{x}_{t+i}^f\right]$$
$$+\sum_{i=1}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)E_t\left[\tilde{\mathbf{x}}_{t+i}^f-\mathbf{x}_{t+i}^f\right]$$
$$+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^2\left(\sigma\boldsymbol{\eta\nu}\otimes\left(\mathbf{h_x}\mathbf{x}_t^s+\tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)$$
$$=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)E_t\left[\tilde{\mathbf{x}}_{t+1}^f\otimes\tilde{\mathbf{x}}_{t+1}^f\otimes\tilde{\mathbf{x}}_{t+1}^f-\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right]$$
$$+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)E_t\left[\tilde{\mathbf{x}}_{t+2}^f\otimes\tilde{\mathbf{x}}_{t+2}^f\otimes\tilde{\mathbf{x}}_{t+2}^f-\mathbf{x}_{t+2}^f\otimes\mathbf{x}_{t+2}^f\otimes\mathbf{x}_{t+2}^f\right]$$
$$+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)E_t\left[\tilde{\mathbf{x}}_{t+1}^f-\mathbf{x}_{t+1}^f\right]$$
$$+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)E_t\left[\tilde{\mathbf{x}}_{t+2}^f-\mathbf{x}_{t+2}^f\right]$$
$$+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^2\left(\sigma\boldsymbol{\eta\nu}\otimes\left(\mathbf{h_x}\mathbf{x}_t^s+\tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)$$

$$=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)X_2+$$
$$+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)E_t\left[\tilde{\mathbf{x}}_{t+2}^f\otimes\tilde{\mathbf{x}}_{t+2}^f\otimes\tilde{\mathbf{x}}_{t+2}^f-\mathbf{x}_{t+2}^f\otimes\mathbf{x}_{t+2}^f\otimes\mathbf{x}_{t+2}^f\right]$$
$$+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)E_t\left[\tilde{\mathbf{x}}_{t+2}^f-\mathbf{x}_{t+2}^f\right]$$

Hence, in general

$$X_k=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)X_{k-1}+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}_{t+k-1}^f\otimes\tilde{\mathbf{x}}_{t+k-1}^f\otimes\tilde{\mathbf{x}}_{t+k-1}^f-\mathbf{x}_{t+k-1}^f\otimes\mathbf{x}_{t+k-1}^f\otimes\mathbf{x}_{t+k-1}^f\right)$$
$$+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\tilde{\mathbf{x}}_{t+k-1}^f-\mathbf{x}_{t+k-1}^f\right)$$

10.3.3 Summarizing

At third order, the total effect on the state variables is:

$$E_t\left[\tilde{\mathbf{x}}_{t+l}-\mathbf{x}_{t+l}\right]=E_t\left[\tilde{\mathbf{x}}_{t+l}^f-\mathbf{x}_{t+l}^f\right]+E_t\left[\tilde{\mathbf{x}}_{t+l}^s-\mathbf{x}_{t+l}^s\right]+E_t\left[\tilde{\mathbf{x}}_{t+l}^{rd}-\mathbf{x}_{t+l}^{rd}\right]$$

For the control variables:

$$\mathbf{y}_{t+l}^{rd}=\mathbf{g_x}\left(\mathbf{x}_{t+l}^f+\mathbf{x}_{t+l}^s+\mathbf{x}_{t+l}^{rd}\right)+\tfrac{1}{2}\mathbf{G_{xx}}\left(\left(\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^f\right)+2\left(\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^s\right)\right)$$
$$+\tfrac{1}{6}\mathbf{G_{xxx}}\left(\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^f\right)+\tfrac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2+\tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+l}^f+\tfrac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$
$$\tilde{\mathbf{y}}_{t+l}^{rd}=\mathbf{g_x}\left(\tilde{\mathbf{x}}_{t+l}^f+\tilde{\mathbf{x}}_{t+l}^s+\tilde{\mathbf{x}}_{t+l}^{rd}\right)+\tfrac{1}{2}\mathbf{G_{xx}}\left(\left(\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f\right)+2\left(\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^s\right)\right)$$
$$+\tfrac{1}{6}\mathbf{G_{xxx}}\left(\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f\right)+\tfrac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2+\tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2\tilde{\mathbf{x}}_{t+l}^f+\tfrac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$

So:

$$E_t\left[\tilde{\mathbf{y}}_{t+l}^{rd}-\mathbf{y}_{t+l}^{rd}\right]$$
$$=\mathbf{g_x}\left(E_t\left[\tilde{\mathbf{x}}_{t+l}^f-\mathbf{x}_{t+l}^f\right]+E_t\left[\tilde{\mathbf{x}}_{t+l}^s-\mathbf{x}_{t+l}^s\right]+E_t\left[\tilde{\mathbf{x}}_{t+l}^{rd}-\mathbf{x}_{t+l}^{rd}\right]\right)$$
$$+\tfrac{1}{2}\mathbf{G_{xx}}\left(E_t\left[\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f-\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^f\right]+2E_t\left[\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^s-\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^s\right]\right)$$
$$+\tfrac{1}{6}\mathbf{G_{xxx}}E_t\left[\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f-\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^f\right]+\tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}_{t+l}^f-\mathbf{x}_{t+l}^f\right]$$

11 Impulse response functions - GIRF

This section derives closed-form solutions for the generalized impulse response function in non-linear DSGE models when defined as the GIRF. That is

$$\begin{aligned} GIRF_{\mathbf{var}}(l, \boldsymbol{\nu}, \mathbf{w}_t) &= E_t\left[\mathbf{var}_{t+l} | \boldsymbol{\epsilon}_{t+1} = \boldsymbol{\nu}, \boldsymbol{\epsilon}_{t+2}, \boldsymbol{\epsilon}_{t+3}, ..., \boldsymbol{\epsilon}_{t+l}\right] \\ & -E_t\left[\mathbf{var}_{t+l} | \boldsymbol{\epsilon}_{t+1} = \boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}_{t+2}, \boldsymbol{\epsilon}_{t+3}, ..., \boldsymbol{\epsilon}_{t+l}\right] \end{aligned}$$

To reduce the notational burden in the derivations below, we adopt the parsimonious notation

$$IRF_{\mathbf{var}}(l, \boldsymbol{\nu}, \mathbf{w}_t) = E_t[\widetilde{\mathbf{var}}_{t+l}] - E_t[\mathbf{var}_{t+l}]$$

in relation to the conditional expectation operators.

11.1 At first order

Recall that we have:

$$\mathbf{x}^f_{t+1} = \mathbf{h_x}\mathbf{x}^f_t + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$

and

$$\begin{aligned} \mathbf{x}^f_{t+2} &= \mathbf{h_x}\mathbf{x}^f_{t+1} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2} \\ &= \mathbf{h_x}\left(\mathbf{h_x}\mathbf{x}^f_t + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2} \\ &= \mathbf{h}^2_{\mathbf{x}}\mathbf{x}^f_t + \mathbf{h_x}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2} \end{aligned}$$

and

$$\begin{aligned} \mathbf{x}^f_{t+3} &= \mathbf{h_x}\mathbf{x}^f_{t+2} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+3} \\ &= \mathbf{h_x}\left(\mathbf{h}^2_{\mathbf{x}}\mathbf{x}^f_t + \mathbf{h_x}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2}\right) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+3} \\ &= \mathbf{h}^3_{\mathbf{x}}\mathbf{x}^f_t + \mathbf{h}^2_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \mathbf{h_x}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+3} \\ &= \mathbf{h}^3_{\mathbf{x}}\mathbf{x}^f_t + \sum_{j=1}^{3}\mathbf{h}^{3-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \end{aligned}$$

In general

$$\mathbf{x}^f_{t+l} = \mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t + \sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

With a shock of $\boldsymbol{\nu}$ in period $t+1$, we have

$$\tilde{\mathbf{x}}^f_{t+l} = \mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t + \sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$

where we define $\boldsymbol{\delta}_t$ such that:

$$\begin{aligned} \boldsymbol{\delta}_{t+j} &= \boldsymbol{\nu} \quad \text{for } j = 1 \\ \boldsymbol{\delta}_{t+j} &= \boldsymbol{\epsilon}_{t+j} \quad \text{for } j \neq 1 \end{aligned}$$

Hence, agents know the size of the shock $\boldsymbol{\nu}$ at time $t+1$, and it is therefore in agents' information set. That we do not need to include more conditional statements on the expectation operator E_t.

So

$$\begin{aligned} E_t\left[\tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l}\right] &= E_t\left[\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} - \sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right] \\ &= \mathbf{h}^{l-1}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \end{aligned}$$

$= \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$
because $\boldsymbol{\delta}_{t+1} = \boldsymbol{\nu}$

and
$E_t\left[\tilde{\mathbf{y}}^f_{t+l} - \mathbf{y}^f_{t+l}\right] = \mathbf{g}_{\mathbf{x}} E_t\left[\tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l}\right]$

11.2 At second order

We need to consider:

$$\mathbf{x}^s_{t+1} = \mathbf{h}_{\mathbf{x}}\mathbf{x}^s_t + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$

$$\begin{aligned}
\mathbf{x}^s_{t+2} &= \mathbf{h}_{\mathbf{x}}\mathbf{x}^s_{t+1} + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1}\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\
&= \mathbf{h}_{\mathbf{x}}\left(\mathbf{h}_{\mathbf{x}}\mathbf{x}^s_t + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1}\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\
&= \mathbf{h}^2_{\mathbf{x}}\mathbf{x}^s_t + \mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1}\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2
\end{aligned}$$

$$\begin{aligned}
\mathbf{x}^s_{t+3} &= \mathbf{h}_{\mathbf{x}}\mathbf{x}^s_{t+2} + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+2} \otimes \mathbf{x}^f_{t+2}\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\
&= \mathbf{h}_{\mathbf{x}}\left(\mathbf{h}^2_{\mathbf{x}}\mathbf{x}^s_t + \mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1}\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \\
&\quad + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+2} \otimes \mathbf{x}^f_{t+2}\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\
&= \mathbf{h}^3_{\mathbf{x}}\mathbf{x}^s_t + \mathbf{h}^2_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \mathbf{h}^2_{\mathbf{x}}\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1}\right) + \mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\
&\quad + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+2} \otimes \mathbf{x}^f_{t+2}\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\
&= \mathbf{h}^3_{\mathbf{x}}\mathbf{x}^s_t + \mathbf{h}^2_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1}\right) + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+2} \otimes \mathbf{x}^f_{t+2}\right) \\
&\quad + \mathbf{h}^2_{\mathbf{x}}\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\
&= \mathbf{h}^3_{\mathbf{x}}\mathbf{x}^s_t + \sum_{j=0}^{2}\mathbf{h}^{2-j}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+j} \otimes \mathbf{x}^f_{t+j}\right) + \left(\sum_{j=0}^{2}\mathbf{h}^{2-j}_{\mathbf{x}}\right)\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2
\end{aligned}$$

and in general

$$\mathbf{x}^s_{t+l} = \mathbf{h}^l_{\mathbf{x}}\mathbf{x}^s_t + \sum_{j=0}^{l-1}\mathbf{h}^{l-1-j}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+j} \otimes \mathbf{x}^f_{t+j}\right) + \left(\sum_{j=0}^{l-1}\mathbf{h}^{l-1-j}_{\mathbf{x}}\right)\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$

for $l = 1, 2, 3, ...$

Thus, to compute $E_t\left[\tilde{\mathbf{x}}^s_{t+l} - \mathbf{x}^s_{t+l}\right]$, we need to find $E_t\left[\tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l} \otimes \mathbf{x}^f_{t+l}\right]$. Hence, consider:

$$\begin{aligned}
\mathbf{x}^f_{t+l} \otimes \mathbf{x}^f_{t+l} &= \left(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t + \sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t + \sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \\
&= \mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t \otimes \mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t + \mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t \otimes \sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \\
&\quad + \sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t + \sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}
\end{aligned}$$

and

$$\tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^f_{t+l} = \mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t \otimes \mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t + \mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t \otimes \sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$

$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$

This means that:
$E_t\left[\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f-\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^f\right]$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$-\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f-\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$-\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f-0$$
$$-0-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$
using $E_t\left[\boldsymbol{\epsilon}_{t+j}\right]=\mathbf{0}$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$
$$+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$
$$-\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$
$$-\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)]$$

$$= E_t[\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + 0$$
$$+0 + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$-\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + 0$$
$$-0 - \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$= E_t[\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+0$$
$$-\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$
$$-0]$$

$$= E_t[\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$-\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}]$$

$$= \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$
$$-E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$$

So we only need to compute $E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$. We then note that $E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$

$$= E_t\left[\left(\mathbf{h}_{\mathbf{x}}^{l-1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\right]$$

using $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}$ if $\mathbf{AC}$ and $\mathbf{BD}$ are defined

$$= \left(\mathbf{h}_{\mathbf{x}}^{l-1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\right) E_t\left[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$$

$$= \left(\mathbf{h}_{\mathbf{x}}^{l-1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\right) \boldsymbol{\Lambda}$$

where $\boldsymbol{\Lambda} \equiv E_t\left[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$. We then have
$\boldsymbol{\Lambda} = E_t\left[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$

$$= E_t\left\{\left\{\sigma\boldsymbol{\eta}\left(\gamma_2,:\right)\boldsymbol{\epsilon}_{t+1}\sigma\boldsymbol{\eta}\left(\gamma_1,:\right)\boldsymbol{\epsilon}_{t+1}\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}$$

$$= E_t\left\{\left\{\sigma\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}\left(\gamma_2,\phi_1\right)\boldsymbol{\epsilon}_{t+1}\left(\phi_1,1\right)\sum_{\phi_2=1}^{n_e}\sigma\boldsymbol{\eta}\left(\gamma_1,\phi_2\right)\boldsymbol{\epsilon}_{t+1}\left(\phi_2,1\right)\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}$$

$$= E_t\left\{\left\{\sigma\sum_{\phi_1=1}^{n_e}\sum_{\phi_2=1}^{n_e}\boldsymbol{\eta}\left(\gamma_2,\phi_1\right)\boldsymbol{\epsilon}_{t+1}\left(\phi_1,1\right)\sigma\boldsymbol{\eta}\left(\gamma_1,\phi_2\right)\boldsymbol{\epsilon}_{t+1}\left(\phi_2,1\right)\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}$$

$$= \left\{\left\{\sigma\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}\left(\gamma_2,\phi_1\right)\sigma\boldsymbol{\eta}\left(\gamma_1,\phi_1\right)\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}$$

$$= \left\{\left\{\sigma^2\boldsymbol{\eta}\left(\gamma_2,:\right)\boldsymbol{\eta}\left(\gamma_1,:\right)'\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}$$

Thus
$E_t\left[\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}-\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\right]=\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^{l-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}+\mathbf{h}^{l-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\mathbf{h}^{l-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}\otimes\mathbf{h}^{l-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}$
$-\left(\mathbf{h}^{l-1}_{\mathbf{x}}\otimes\mathbf{h}^{l-1}_{\mathbf{x}}\right)\boldsymbol{\Lambda}$

$=\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^{l-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}+\mathbf{h}^{l-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\left(\mathbf{h}^{l-1}_{\mathbf{x}}\otimes\mathbf{h}^{l-1}_{\mathbf{x}}\right)(\sigma\boldsymbol{\eta\nu}\otimes\sigma\boldsymbol{\eta\nu})$
$-\left(\mathbf{h}^{l-1}_{\mathbf{x}}\otimes\mathbf{h}^{l-1}_{\mathbf{x}}\right)\boldsymbol{\Lambda}$

$=\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^{l-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}+\mathbf{h}^{l-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\left(\mathbf{h}^{l-1}_{\mathbf{x}}\otimes\mathbf{h}^{l-1}_{\mathbf{x}}\right)((\sigma\boldsymbol{\eta\nu}\otimes\sigma\boldsymbol{\eta\nu})-\boldsymbol{\Lambda})$

Or (using another index)
$E_t\left[\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^f_{t+j}-\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right]=\mathbf{h}^j_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^{j-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}+\mathbf{h}^{j-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}\otimes\mathbf{h}^j_{\mathbf{x}}\mathbf{x}^f_t+\left(\mathbf{h}^{j-1}_{\mathbf{x}}\otimes\mathbf{h}^{j-1}_{\mathbf{x}}\right)((\sigma\boldsymbol{\eta\nu}\otimes\sigma\boldsymbol{\eta\nu})-\boldsymbol{\Lambda})$
for $j=1,2,3,...$

Thus, we have in general
$E_t\left[\tilde{\mathbf{x}}^s_{t+l}-\mathbf{x}^s_{t+l}\right]=E_t\left[\sum_{j=0}^{l-1}\mathbf{h}^{l-1-j}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^f_{t+j}\right)-\sum_{j=0}^{l-1}\mathbf{h}^{l-1-j}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right)\right]$

$=\sum_{j=1}^{l-1}\mathbf{h}^{l-1-j}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^f_{t+j}-\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right]$
the shock hits in period $t+1$, so $\left(\tilde{\mathbf{x}}^f_t\otimes\tilde{\mathbf{x}}^f_t\right)=\mathbf{x}^f_t\otimes\mathbf{x}^f_t$

$=\sum_{j=1}^{l-1}\mathbf{h}^{l-1-j}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{h}^j_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^{j-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}+\mathbf{h}^{j-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}\otimes\mathbf{h}^j_{\mathbf{x}}\mathbf{x}^f_t+\left(\mathbf{h}^{j-1}_{\mathbf{x}}\otimes\mathbf{h}^{j-1}_{\mathbf{x}}\right)((\sigma\boldsymbol{\eta\nu}\otimes\sigma\boldsymbol{\eta\nu})-\boldsymbol{\Lambda})\right)$

If we restrict the focus and do the GIRF's at the unconditional mean of $\mathbf{x}^f_t=\mathbf{0}$, then we get

$$E_t\left[\tilde{\mathbf{x}}^s_{t+l}-\mathbf{x}^s_{t+l}\right]=\sum_{j=1}^{l-1}\mathbf{h}^{l-1-j}_{\mathbf{x}}\frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\left(\mathbf{h}^{j-1}_{\mathbf{x}}\otimes\mathbf{h}^{j-1}_{\mathbf{x}}\right)(\sigma\boldsymbol{\eta\nu}\otimes\sigma\boldsymbol{\eta\nu}-\boldsymbol{\Lambda})\right)$$

When implementing the GIRF, it may be useful to have a recursive expression. Here, it is must convenient to use
$E_t\left[\tilde{\mathbf{x}}^s_{t+l}-\mathbf{x}^s_{t+l}\right]=\sum_{j=1}^{l-1}\mathbf{h}^{l-1-j}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^f_{t+j}-\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right]$
So
$E_t\left[\tilde{\mathbf{x}}^s_{t+1}-\mathbf{x}^s_{t+1}\right]=0$

$E_t\left[\tilde{\mathbf{x}}^s_{t+2}-\mathbf{x}^s_{t+2}\right]=\sum_{j=1}^{1}\mathbf{h}^{1-j}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^f_{t+j}-\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right]$
$=\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+1}\otimes\tilde{\mathbf{x}}^f_{t+1}-\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right]$

$E_t\left[\tilde{\mathbf{x}}^s_{t+3}-\mathbf{x}^s_{t+3}\right]=\sum_{j=1}^{2}\mathbf{h}^{2-j}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^f_{t+j}-\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right]$
$=\mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+1}\otimes\tilde{\mathbf{x}}^f_{t+1}-\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right]$
$+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+2}\otimes\tilde{\mathbf{x}}^f_{t+2}-\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right]$
$=\mathbf{h}_{\mathbf{x}}E_t\left[\tilde{\mathbf{x}}^s_{t+2}-\mathbf{x}^s_{t+2}\right]+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+2}\otimes\tilde{\mathbf{x}}^f_{t+2}-\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right]$

So in general
$E_t\left[\tilde{\mathbf{x}}^s_{t+k}-\mathbf{x}^s_{t+k}\right]=\mathbf{h_x}E_t\left[\tilde{\mathbf{x}}^s_{t+k-1}-\mathbf{x}^s_{t+k-1}\right]+\frac{1}{2}\mathbf{H_{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+k-1}\otimes\tilde{\mathbf{x}}^f_{t+k-1}-\mathbf{x}^f_{t+k-1}\otimes\mathbf{x}^f_{t+k-1}\right]$

For the total state variable:
$E_t\left[\tilde{\mathbf{x}}_{t+l}-\mathbf{x}_{t+l}\right]=E_t\left[\tilde{\mathbf{x}}^f_{t+l}-\mathbf{x}^f_{t+l}\right]+E_t\left[\tilde{\mathbf{x}}^s_{t+l}-\mathbf{x}^s_{t+l}\right]$

For the control variables:
$\mathbf{y}^s_{t+l}=\mathbf{g_x}\left(\mathbf{x}^f_{t+l}+\mathbf{x}^s_{t+l}\right)+\frac{1}{2}\mathbf{G_{xx}}\left(\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\right)+\frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$
$\tilde{\mathbf{y}}^s_{t+l}=\mathbf{g_x}\left(\tilde{\mathbf{x}}^f_{t+l}+\tilde{\mathbf{x}}^s_{t+l}\right)+\frac{1}{2}\mathbf{G_{xx}}\left(\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}\right)+\frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$

$E_t\left[\tilde{\mathbf{y}}^s_{t+l}-\mathbf{y}^s_{t+l}\right]=\mathbf{g_x}\left(E_t\left[\tilde{\mathbf{x}}^f_{t+l}-\mathbf{x}^f_{t+l}\right]+E_t\left[\tilde{\mathbf{x}}^s_{t+l}-\mathbf{x}^s_{t+l}\right]\right)+\frac{1}{2}\mathbf{G_{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}-\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\right]$

11.3 At third order

At third order, we additionally need to consider:
$\mathbf{x}^{rd}_{t+1}=\mathbf{h_x}\mathbf{x}^{rd}_t+\mathbf{H_{xx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)+\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_t+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$

and
$\mathbf{x}^{rd}_{t+2}=\mathbf{h_x}\mathbf{x}^{rd}_{t+1}+\mathbf{H_{xx}}\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^s_{t+1}\right)+\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right)+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_{t+1}+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$

$=\mathbf{h_x}\left(\mathbf{h_x}\mathbf{x}^{rd}_t+\mathbf{H_{xx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)+\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_t+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\right)$
$+\mathbf{H_{xx}}\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^s_{t+1}\right)+\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right)+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_{t+1}+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$

$=\mathbf{h}^2_\mathbf{x}\mathbf{x}^{rd}_t+\mathbf{h_x}\mathbf{H_{xx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)+\mathbf{h_x}\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)+\mathbf{h_x}\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_t+\mathbf{h_x}\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$
$+\mathbf{H_{xx}}\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^s_{t+1}\right)+\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right)+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_{t+1}+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$

and
$\mathbf{x}^{rd}_{t+3}=\mathbf{h_x}\mathbf{x}^{rd}_{t+2}+\mathbf{H_{xx}}\left(\mathbf{x}^f_{t+2}\otimes\mathbf{x}^s_{t+2}\right)+\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right)+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_{t+2}+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$

$=\mathbf{h_x}[\mathbf{h}^2_\mathbf{x}\mathbf{x}^{rd}_t+\mathbf{h_x}\mathbf{H_{xx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)+\mathbf{h_x}\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)+\mathbf{h_x}\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_t+\mathbf{h_x}\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$
$+\mathbf{H_{xx}}\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^s_{t+1}\right)+\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right)+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_{t+1}+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3]$
$+\mathbf{H_{xx}}\left(\mathbf{x}^f_{t+2}\otimes\mathbf{x}^s_{t+2}\right)+\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right)+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_{t+2}+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$

$=\mathbf{h}^3_\mathbf{x}\mathbf{x}^{rd}_t+\mathbf{h}^2_\mathbf{x}\mathbf{H_{xx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)+\mathbf{h}^2_\mathbf{x}\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)+\mathbf{h}^2_\mathbf{x}\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_t+\mathbf{h}^2_\mathbf{x}\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$
$+\mathbf{h_x}\mathbf{H_{xx}}\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^s_{t+1}\right)+\mathbf{h_x}\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right)+\mathbf{h_x}\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_{t+1}+\mathbf{h_x}\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$
$+\mathbf{H_{xx}}\left(\mathbf{x}^f_{t+2}\otimes\mathbf{x}^s_{t+2}\right)+\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right)+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_{t+2}+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$

$=\mathbf{h}^3_\mathbf{x}\mathbf{x}^{rd}_t+\sum_{j=0}^{2}\mathbf{h}^{2-j}_\mathbf{x}\mathbf{H_{xx}}\left(\mathbf{x}^f_{t+j}\otimes\mathbf{x}^s_{t+j}\right)$
$+\sum_{j=0}^{2}\mathbf{h}^{2-j}_\mathbf{x}\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right)$

$$+\sum_{j=0}^{2}\mathbf{h}_{\mathbf{x}}^{2-j}\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+j}^{f}$$
$$+\sum_{j=0}^{2}\mathbf{h}_{\mathbf{x}}^{2-j}\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

$$=\mathbf{h}_{\mathbf{x}}^{3}\mathbf{x}_{t}^{rd}+\sum_{j=0}^{2}\mathbf{h}_{\mathbf{x}}^{2-j}\left[\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{s}\right)+\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}\left(\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\right)+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+j}^{f}\right]$$
$$+\left(\sum_{j=0}^{2}\mathbf{h}_{\mathbf{x}}^{2-j}\right)\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

In general

$$\mathbf{x}_{t+l}^{rd}=\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{rd}+\sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\left[\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{s}\right)+\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}\left(\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\right)+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+j}^{f}\right]$$
$$+\left(\sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\right)\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

Thus

$$E_t\left[\tilde{\mathbf{x}}_{t+l}^{rd}-\mathbf{x}_{t+l}^{rd}\right]=E_t[\sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\left[\mathbf{H}_{\mathbf{xx}}\left(\tilde{\mathbf{x}}_{t+j}^{f}\otimes\tilde{\mathbf{x}}_{t+j}^{s}\right)+\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}\left(\tilde{\mathbf{x}}_{t+j}^{f}\otimes\tilde{\mathbf{x}}_{t+j}^{f}\otimes\tilde{\mathbf{x}}_{t+j}^{f}\right)+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\tilde{\mathbf{x}}_{t+j}^{f}\right]$$
$$-\{\sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\left[\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{s}\right)+\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}\left(\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\right)+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+j}^{f}\right]\}]$$

$$=\sum_{j=1}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\left(\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}_{t+j}^{f}\otimes\tilde{\mathbf{x}}_{t+j}^{s}-\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{s}\right]+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}_{t+j}^{f}-\mathbf{x}_{t+j}^{f}\right]\right)$$
$$+\sum_{j=1}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}_{t+j}^{f}\otimes\tilde{\mathbf{x}}_{t+j}^{f}\otimes\tilde{\mathbf{x}}_{t+j}^{f}-\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\right]$$

as the shock hits the economy in period $t+1$

A recursive version:

$$E_t\left[\tilde{\mathbf{x}}_{t+1}^{rd}-\mathbf{x}_{t+1}^{rd}\right]=0$$

$$E_t\left[\tilde{\mathbf{x}}_{t+2}^{rd}-\mathbf{x}_{t+2}^{rd}\right]=\sum_{j=1}^{1}\mathbf{h}_{\mathbf{x}}^{1-j}\left(\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}_{t+j}^{f}\otimes\tilde{\mathbf{x}}_{t+j}^{s}-\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{s}\right]+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}_{t+j}^{f}-\mathbf{x}_{t+j}^{f}\right]\right)$$
$$+\sum_{j=1}^{1}\mathbf{h}_{\mathbf{x}}^{1-j}\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}_{t+j}^{f}\otimes\tilde{\mathbf{x}}_{t+j}^{f}\otimes\tilde{\mathbf{x}}_{t+j}^{f}-\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\right]$$
$$=\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}_{t+1}^{f}\otimes\tilde{\mathbf{x}}_{t+1}^{s}-\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{s}\right]+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}_{t+1}^{f}-\mathbf{x}_{t+1}^{f}\right]$$
$$+\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}_{t+1}^{f}\otimes\tilde{\mathbf{x}}_{t+1}^{f}\otimes\tilde{\mathbf{x}}_{t+1}^{f}-\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right]$$

$$E_t\left[\tilde{\mathbf{x}}_{t+3}^{rd}-\mathbf{x}_{t+3}^{rd}\right]=\sum_{j=1}^{2}\mathbf{h}_{\mathbf{x}}^{2-j}\left(\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}_{t+j}^{f}\otimes\tilde{\mathbf{x}}_{t+j}^{s}-\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{s}\right]+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}_{t+j}^{f}-\mathbf{x}_{t+j}^{f}\right]\right)$$
$$+\sum_{j=1}^{2}\mathbf{h}_{\mathbf{x}}^{2-j}\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}_{t+j}^{f}\otimes\tilde{\mathbf{x}}_{t+j}^{f}\otimes\tilde{\mathbf{x}}_{t+j}^{f}-\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\right]$$
$$=\mathbf{h}_{\mathbf{x}}\left(\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}_{t+1}^{f}\otimes\tilde{\mathbf{x}}_{t+1}^{s}-\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{s}\right]+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}_{t+1}^{f}-\mathbf{x}_{t+1}^{f}\right]\right)$$
$$+\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}_{t+2}^{f}\otimes\tilde{\mathbf{x}}_{t+2}^{s}-\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{s}\right]+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}_{t+2}^{f}-\mathbf{x}_{t+2}^{f}\right]$$
$$+\mathbf{h}_{\mathbf{x}}\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}_{t+1}^{f}\otimes\tilde{\mathbf{x}}_{t+1}^{f}\otimes\tilde{\mathbf{x}}_{t+1}^{f}-\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right]$$

$$+\frac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}^f_{t+2}\otimes\tilde{\mathbf{x}}^f_{t+2}\otimes\tilde{\mathbf{x}}^f_{t+2}-\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right]$$

$$=\mathbf{h}_{\mathbf{x}}E_t\left[\tilde{\mathbf{x}}^{rd}_{t+2}-\mathbf{x}^{rd}_{t+2}\right]$$
$$+\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+2}\otimes\tilde{\mathbf{x}}^s_{t+2}-\mathbf{x}^f_{t+2}\otimes\mathbf{x}^s_{t+2}\right]+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}^f_{t+2}-\mathbf{x}^f_{t+2}\right]$$
$$+\frac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}^f_{t+2}\otimes\tilde{\mathbf{x}}^f_{t+2}\otimes\tilde{\mathbf{x}}^f_{t+2}-\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right]$$

So in general

$$E_t\left[\tilde{\mathbf{x}}^{rd}_{t+k}-\mathbf{x}^{rd}_{t+k}\right]=\mathbf{h}_{\mathbf{x}}E_t\left[\tilde{\mathbf{x}}^{rd}_{t+k-1}-\mathbf{x}^{rd}_{t+k-1}\right]+\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+k-1}\otimes\tilde{\mathbf{x}}^s_{t+k-1}-\mathbf{x}^f_{t+k-1}\otimes\mathbf{x}^s_{t+k-1}\right]$$
$$+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}^f_{t+k-1}-\mathbf{x}^f_{t+k-1}\right]$$
$$+\frac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}^f_{t+k-1}\otimes\tilde{\mathbf{x}}^f_{t+k-1}\otimes\tilde{\mathbf{x}}^f_{t+k-1}-\mathbf{x}^f_{t+k-1}\otimes\mathbf{x}^f_{t+k-1}\otimes\mathbf{x}^f_{t+k-1}\right]$$

Thus, we know $E_t\left[\tilde{\mathbf{x}}^f_{t+j}-\mathbf{x}^f_{t+j}\right]$. So we only need to compute $E_t\left[\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^f_{t+j}-\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right]$ and $E_t\left[\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^s_{t+j}-\mathbf{x}^f_{t+j}\otimes\mathbf{x}^s_{t+j}\right]$. This is done in the next two subsections. For these derivations recall that we define $\boldsymbol{\delta}_t$ such that:

$$\boldsymbol{\delta}_{t+j}=\boldsymbol{\nu}\quad\text{for } j=1$$
$$\boldsymbol{\delta}_{t+j}=\boldsymbol{\epsilon}_{t+j}\quad\text{for } j\neq 1$$

11.3.1 For $\left(\mathbf{x}^f_t\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)$

Consider:

$$\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}=\left(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\left(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\left(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$

$$=(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j})$$
$$\otimes\left(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$

$$=(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j})\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t$$
$$+(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j})\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

$$=\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t$$
$$+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t$$
$$+\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

And we therefore have

$$\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}=\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t$$

$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$

This means that:
$E_t\left[\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f-\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^f\right]$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$

$$-E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$=E_t[0+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$

$$-E_t[+\mathbf{0}+\mathbf{0}$$
$$+\mathbf{0}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{0}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

using $E_t\left[\boldsymbol{\epsilon}_{t+i}\right]=0$ and cancelling the term $\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$

$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$

$$-E_t[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}-\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

We now evaluate the expressions on each of the four last lines:

$$A_1\equiv E_t\left[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right]$$

$$=E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$-\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$-\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f]$$

$$= E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+ \left(\sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$- \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$- \left(\sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f]$$

$$= E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \mathbf{0}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+ \left(\mathbf{0} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$- \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \mathbf{0}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$- \left(\mathbf{0} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f]$$

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+$$
$$-\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$-]$$

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f - \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f]$$

$$= \left(\mathbf{h}_{\mathbf{x}}^{l-1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f - E_t\left[\left(\mathbf{h}_{\mathbf{x}}^{l-1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right]$$

using $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}$ if $\mathbf{AC}$ and $\mathbf{BD}$ are defined

$$= \left(\mathbf{h}_{\mathbf{x}}^{l-1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f - \left(\mathbf{h}_{\mathbf{x}}^{l-1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\right) E_t\left[\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\right] \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$

$$= \left(\mathbf{h}_{\mathbf{x}}^{l-1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f - \left(\mathbf{h}_{\mathbf{x}}^{l-1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\right) \boldsymbol{\Lambda} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$

$$= \left(\mathbf{h}_{\mathbf{x}}^{l-1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\right)\left(\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) - \boldsymbol{\Lambda}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$

Note $A_1 \equiv E_t\left[\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f - \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right]$

Therefore we immediately see from the structure of the terms that
$A_2 \equiv \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} - \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} - \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}]$$

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) - \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)]$$

$$= \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\right)\left(\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) - \boldsymbol{\Lambda}\right)$$

For the third term:

$$A_3 \equiv \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} - \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} - \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}]$$

$$= \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} - \boldsymbol{\Gamma}(l)$$

The only element which is not directly computable is $\Gamma \equiv E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$ which must be computed element by element. Hence, consider

$$\boldsymbol{\Gamma}(l) \equiv E_t\left[\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \left\{\mathbf{h}_{\mathbf{x}}^{l}(\gamma_2,:)\,\mathbf{x}_t^f \times \left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\,\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right]$$

$$= E_t\left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_{3,:})\,\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \times \left\{\mathbf{h}_{\mathbf{x}}^{l}(\gamma_2,:)\,\mathbf{x}_t^f \times \left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\,\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}$$

$$= E_t\left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_{3,:})\,\sigma\sum_{\phi_2=1}^{n_e}\boldsymbol{\eta}(:,\phi_2)\,\boldsymbol{\epsilon}_{t+1}(\phi_2,1) \times \left\{\mathbf{h}_{\mathbf{x}}^{l}(\gamma_2,:)\,\mathbf{x}_t^f \times \left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\,\sigma\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}(:,\phi_1)\,\boldsymbol{\epsilon}_{t+1}(\phi_1,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}$$

$$= E_t\left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_{3,:})\,\sigma\sum_{\phi_2=1}^{n_e}\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}(:,\phi_2)\,\boldsymbol{\epsilon}_{t+1}(\phi_2,1) \times \left\{\mathbf{h}_{\mathbf{x}}^{l}(\gamma_2,:)\,\mathbf{x}_t^f \times \left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\,\sigma\boldsymbol{\eta}(:,\phi_1)\,\boldsymbol{\epsilon}_{t+1}(\phi_1,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}$$

$$= \left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_{3,:})\,\sigma\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}(:,\phi_1) \times \left\{\mathbf{h}_{\mathbf{x}}^{l}(\gamma_2,:)\,\mathbf{x}_t^f \times \left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\,\sigma\boldsymbol{\eta}(:,\phi_1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}$$

because the innovations are indepdendent. This expression is directly implementable.

$$A_4 = E_t\left[\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} - \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]$$

$$= E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)$$
$$- \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)]$$

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)$$
$$+ \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)$$
$$-\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$
$$- \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)]$$

$$= E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)$$

$$+\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$
$$-\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)]$$

$$=E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$-\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$
$$-\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$=E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{0}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+\left(\mathbf{0}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)$$
$$+\left(\mathbf{0}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$+\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\mathbf{0}\right)$$
$$-(\mathbf{0}+\mathbf{0})$$
$$-(0+0)$$
$$-\left(0+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$

$$+\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}+0\right)$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)$$
$$+(0)]$$

$$=\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$
$$+E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right]$$
$$+E_t\left[\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right]$$
$$+E_t\left[\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right]$$
$$-E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$$

How consider the following terms:

$$A_{4,1}\equiv E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-2}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-3}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+3}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+...$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+l}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-2}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{h}_{\mathbf{x}}^{l-2}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2}$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-3}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{h}_{\mathbf{x}}^{l-3}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+3}$$
$$+...$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+l}\otimes\mathbf{h}_{\mathbf{x}}^{l-l}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+l}]$$

because the innovations are independent across time

$$=\sum_{j=2}^{l}E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]$$

$$=\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes E_t\left[\left(\mathbf{h}_{\mathbf{x}}^{l-j}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\right]$$

$$=\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-j}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\right)E_t\left[\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\right]$$

because the innovations are identical distributed across time

$$=\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-j}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\right)\Lambda$$

We therefore immediately have for the second term:

$$A_{4,2}\equiv E_t\left[\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right]=\sum_{j=2}^{l}\Lambda\left(\mathbf{h}_{\mathbf{x}}^{l-j}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$

For the third term (when using the results from above)

$$A_{4,3} \equiv E_t \left[\sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+j} \right]$$

$$= \sum_{j=2}^{l} E_t \left[\mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \right]$$

$$= \sum_{j=2}^{l} \boldsymbol{\Omega}_j$$

where $\boldsymbol{\Omega}_j \equiv E_t \left[\mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \right]$. So

$$\boldsymbol{\Omega}_j \equiv E_t \left[\mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \otimes \left\{ \mathbf{h}_{\mathbf{x}}^{l-1} (\gamma_2, :) \sigma \boldsymbol{\eta} \boldsymbol{\nu} \times \left\{ \mathbf{h}_{\mathbf{x}}^{l-j} (\gamma_1, :) \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \right\}_{\gamma_1=1}^{n_x} \right\}_{\gamma_2=1}^{n_x} \right]$$

$$= E_t \left\{ \mathbf{h}_{\mathbf{x}}^{l-j} (\gamma_{3,:}) \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \times \left\{ \mathbf{h}_{\mathbf{x}}^{l-1} (\gamma_2, :) \sigma \boldsymbol{\eta} \boldsymbol{\nu} \times \left\{ \mathbf{h}_{\mathbf{x}}^{l-j} (\gamma_1, :) \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \right\}_{\gamma_1=1}^{n_x} \right\}_{\gamma_2=1}^{n_x} \right\}_{\gamma_3=1}^{n_x}$$

$$= E_t \left\{ \mathbf{h}_{\mathbf{x}}^{l-j} (\gamma_{3,:}) \sigma \textstyle\sum_{\phi_2=1}^{n_e} \boldsymbol{\eta} (:, \phi_2) \boldsymbol{\epsilon}_{t+1} (\phi_2, 1) \times \left\{ \mathbf{h}_{\mathbf{x}}^{l-1} (\gamma_2, :) \sigma \boldsymbol{\eta} \boldsymbol{\nu} \times \left\{ \mathbf{h}_{\mathbf{x}}^{l-j} (\gamma_1, :) \sigma \sum_{\phi_1=1}^{n_e} \boldsymbol{\eta} (:, \phi_1) \boldsymbol{\epsilon}_{t+1} (\phi_1, 1) \right\}_{\gamma_1=1}^{n_x} \right\}_{\gamma_2=1}^{n_x} \right\}_{\gamma_3=}^{n_x}$$

$$= E_t \left\{ \mathbf{h}_{\mathbf{x}}^{l-j} (\gamma_{3,:}) \sigma \textstyle\sum_{\phi_2=1}^{n_e} \sum_{\phi_1=1}^{n_e} \boldsymbol{\eta} (:, \phi_2) \boldsymbol{\epsilon}_{t+1} (\phi_2, 1) \times \left\{ \mathbf{h}_{\mathbf{x}}^{l-1} (\gamma_2, :) \sigma \boldsymbol{\eta} \boldsymbol{\nu} \times \left\{ \mathbf{h}_{\mathbf{x}}^{l-j} (\gamma_1, :) \sigma \boldsymbol{\eta} (:, \phi_1) \boldsymbol{\epsilon}_{t+1} (\phi_1, 1) \right\}_{\gamma_1=1}^{n_x} \right\}_{\gamma_2=1}^{n_x} \right\}_{\gamma_3=1}^{n_x}$$

$$= \left\{ \mathbf{h}_{\mathbf{x}}^{l-j} (\gamma_{3,:}) \sigma \textstyle\sum_{\phi_1=1}^{n_e} \boldsymbol{\eta} (:, \phi_1) \times \left\{ \mathbf{h}_{\mathbf{x}}^{l-1} (\gamma_2, :) \sigma \boldsymbol{\eta} \boldsymbol{\nu} \times \left\{ \mathbf{h}_{\mathbf{x}}^{l-j} (\gamma_1, :) \sigma \boldsymbol{\eta} (:, \phi_1) \right\}_{\gamma_1=1}^{n_x} \right\}_{\gamma_2=1}^{n_x} \right\}_{\gamma_3=1}^{n_x}$$

because the innovations are indepdendent. This expression is directly implementable.

For the fourth term must be computed element by element

$$A_{4,4} \equiv E_t \left[\mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \right]$$

$$= E_t \left[\mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \otimes \left\{ \mathbf{h}_{\mathbf{x}}^{l-1} (\gamma_2, :) \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \times \left\{ \mathbf{h}_{\mathbf{x}}^{l-1} (\gamma_1, :) \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \right\}_{\gamma_1=1}^{n_x} \right\}_{\gamma_2=1}^{n_x} \right]$$

$$= E_t \left[\left\{ \mathbf{h}_{\mathbf{x}}^{l-1} (\gamma_{3,:}) \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \times \left\{ \mathbf{h}_{\mathbf{x}}^{l-1} (\gamma_2, :) \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \times \left\{ \mathbf{h}_{\mathbf{x}}^{l-1} (\gamma_1, :) \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \right\}_{\gamma_1=1}^{n_x} \right\}_{\gamma_2=1}^{n_x} \right\}_{\gamma_3=1}^{n_x} \right]$$

$$= E_t [\{ \mathbf{h}_{\mathbf{x}}^{l-1} (\gamma_{3,:}) \sigma \textstyle\sum_{\phi_3=1}^{n_e} \boldsymbol{\eta} (:, \phi_3) \boldsymbol{\epsilon}_{t+1} (\phi_3, 1) \times$$
$$\{ \mathbf{h}_{\mathbf{x}}^{l-1} (\gamma_2, :) \sigma \textstyle\sum_{\phi_3=1}^{n_e} \boldsymbol{\eta} (:, \phi_2) \boldsymbol{\epsilon}_{t+1} (\phi_2, 1) \times \left\{ \mathbf{h}_{\mathbf{x}}^{l-1} (\gamma_1, :) \sigma \sum_{\phi_1=1}^{n_e} \boldsymbol{\eta} (:, \phi_1) \boldsymbol{\epsilon}_{t+1} (\phi_1, 1) \right\}_{\gamma_1=1}^{n_x} \}_{\gamma_2=1}^{n_x} \}_{\gamma_3=1}^{n_x}]$$

$$= E_t [\{ \mathbf{h}_{\mathbf{x}}^{l-1} (\gamma_{3,:}) \sigma \textstyle\sum_{\phi_1=1}^{n_e} \boldsymbol{\eta} (:, \phi_1) \boldsymbol{\epsilon}_{t+1} (\phi_1, 1) \times$$
$$\{ \mathbf{h}_{\mathbf{x}}^{l-1} (\gamma_2, :) \sigma \boldsymbol{\eta} (:, \phi_1) \boldsymbol{\epsilon}_{t+1} (\phi_1, 1) \times \{ \mathbf{h}_{\mathbf{x}}^{l-1} (\gamma_1, :) \sigma \boldsymbol{\eta} (:, \phi_1) \boldsymbol{\epsilon}_{t+1} (\phi_1, 1) \}_{\gamma_1=1}^{n_x} \}_{\gamma_2=1}^{n_x} \}_{\gamma_3=1}^{n_x}]$$

$$= \{ \sigma^3 \mathbf{h}_{\mathbf{x}}^{l-1} (\gamma_{3,:}) \textstyle\sum_{\phi_1=1}^{n_e} \boldsymbol{\eta} (:, \phi_1) E_t \left[\boldsymbol{\epsilon}_{t+1}^3 (\phi_1, 1) \right] \times \{ \mathbf{h}_{\mathbf{x}}^{l-1} (\gamma_2, :) \boldsymbol{\eta} (:, \phi_1) \times \{ \mathbf{h}_{\mathbf{x}}^{l-1} (\gamma_1, :) \boldsymbol{\eta} (:, \phi_1) \}_{\gamma_1=1}^{n_x} \}_{\gamma_2=1}^{n_x} \}_{\gamma_3=1}^{n_x}]$$

Thus, we finally have:

$$E_t\left[\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}-\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\right]$$

$$
\begin{aligned}
&= \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
A_1)\quad &+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\right)\left((\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\nu})-\boldsymbol{\Lambda}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
A_2)\quad &+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\right)\left((\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\nu})-\boldsymbol{\Lambda}\right)\\
A_3)\quad &+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}-\boldsymbol{\Gamma}(l)\\
A_4)\quad &+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
A_{4,1})\quad &+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-j}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\right)\Lambda\\
A_{4,2})\quad &+\sum_{j=2}^{l}\Lambda\left(\mathbf{h}_{\mathbf{x}}^{l-j}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
A_{4,3})\quad &+\sum_{j=2}^{l}\boldsymbol{\Omega}_j\\
A_{4,4})\quad &+E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]
\end{aligned}
$$

11.3.2 For $\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)$

Recall from above that

$$\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^s=\left(\mathbf{h}_{\mathbf{x}}\mathbf{x}_t^f+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}\mathbf{x}_t^s+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)$$

$$
\begin{aligned}
&=(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_t^f\\
&+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\mathbf{x}_{t+2}^f\otimes\mathbf{x}_{t+2}^s&=(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})\left(\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^s\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_{t+1}^f\\
&+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+2}
\end{aligned}
$$

$$
\begin{aligned}
&=(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})\left[(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_t^f\right.\\
&\quad\left.+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\right]\\
&+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_{t+1}^f\\
&+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+2}
\end{aligned}
$$

$$
\begin{aligned}
&=(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})^2\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)+(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_t^f\\
&+(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)+(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\\
&+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_{t+1}^f\\
&\quad+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+2}
\end{aligned}
$$

and

$$\mathbf{x}_{t+3}^f\otimes\mathbf{x}_{t+3}^s=(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})\left(\mathbf{x}_{t+2}^f\otimes\mathbf{x}_{t+2}^s\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\mathbf{x}_{t+2}^f\otimes\mathbf{x}_{t+2}^f\otimes\mathbf{x}_{t+2}^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_{t+2}^f$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}^s_{t+2}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+3}$$

$$\begin{aligned}
=&\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)[\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^2\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}^f_t\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}^f_t\\
&+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}^s_t\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\\
&+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right)+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}^f_{t+1}\\
&+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}^s_{t+1}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+2}]\\
&+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right)+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}^f_{t+2}\\
&+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}^s_{t+2}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+3}
\end{aligned}$$

$$\begin{aligned}
=&\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^3\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^2\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}^f_t\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^2\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}^f_t\\
&+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^2\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}^s_t\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^2\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^2\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\\
&+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}^f_{t+1}\\
&+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}^s_{t+1}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+2}
\end{aligned}$$

$$\begin{aligned}
&+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right)+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}^f_{t+2}\\
&+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}^s_{t+2}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+3}
\end{aligned}$$

$$\begin{aligned}
=&\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^3\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)+\sum_{i=0}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)\\
&+\sum_{i=0}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}^f_{t+i}\\
&+\sum_{i=0}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1+i}\\
&+\sum_{i=0}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^s_{t+i}\right)\\
&+\sum_{i=0}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)
\end{aligned}$$

And in general

$$\begin{aligned}
\mathbf{x}^f_{t+l}\otimes\mathbf{x}^s_{t+l}=&\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^l\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)\\
&+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}^f_{t+i}\\
&+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1+i}\\
&+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^s_{t+i}\right)\\
&+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)
\end{aligned}$$

for $l=0,1,2,3....$ Note for $l=0$ we have $\mathbf{x}^f_t\otimes\mathbf{x}^s_t=\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)$ as desired.

We therefore have

$$\tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^s_{t+l} = (\mathbf{h_x} \otimes \mathbf{h_x})^l \left(\tilde{\mathbf{x}}^f_t \otimes \tilde{\mathbf{x}}^s_t\right) + \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i}\right)$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \tilde{\mathbf{x}}^f_{t+i}$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \boldsymbol{\delta}_{t+1+i}$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right) \left(\boldsymbol{\delta}_{t+1+i} \otimes \tilde{\mathbf{x}}^s_{t+i}\right)$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\boldsymbol{\delta}_{t+1+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i}\right)$$

Thus
$E_t\left[\tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^s_{t+l} - \mathbf{x}^f_{t+l} \otimes \mathbf{x}^s_{t+l}\right]$

$$= E_t[(\mathbf{h_x} \otimes \mathbf{h_x})^l \left(\tilde{\mathbf{x}}^f_t \otimes \tilde{\mathbf{x}}^s_t\right) + \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i}\right)$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \tilde{\mathbf{x}}^f_{t+i}$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \boldsymbol{\delta}_{t+1+i}$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right) \left(\boldsymbol{\delta}_{t+1+i} \otimes \tilde{\mathbf{x}}^s_{t+i}\right)$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\boldsymbol{\delta}_{t+1+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i}\right)$$

$$-\{(\mathbf{h_x} \otimes \mathbf{h_x})^l \left(\mathbf{x}^f_t \otimes \mathbf{x}^s_t\right) + \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right)$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \mathbf{x}^f_{t+i}$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \boldsymbol{\epsilon}_{t+1+i}$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right) \left(\boldsymbol{\epsilon}_{t+1+i} \otimes \mathbf{x}^s_{t+i}\right)$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\boldsymbol{\epsilon}_{t+1+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right)\}$$
$$]$$

$$= E_t[\sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right)$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i}\right)$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\boldsymbol{\delta}_{t+1+i} - \boldsymbol{\epsilon}_{t+1+i}\right)$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right) \left(\boldsymbol{\delta}_{t+1+i} \otimes \tilde{\mathbf{x}}^s_{t+i} - \boldsymbol{\epsilon}_{t+1+i} \otimes \mathbf{x}^s_{t+i}\right)$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\boldsymbol{\delta}_{t+1+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} - \boldsymbol{\epsilon}_{t+1+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right)]$$

because the shock hits in period $t+1$, meing that $\mathbf{x}^f_t = \tilde{\mathbf{x}}^f_t$ and similar for $\tilde{\mathbf{x}}^s_t$

$$= E_t[\sum_{i=0}^{l-1} (\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i} \left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)$$
$$+\sum_{i=0}^{l-1} (\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i} \left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\right)$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\delta}_{t+1}$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\delta}_{t+1}\otimes\tilde{\mathbf{x}}^s_t-\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}^s_t\right)$$
$$+\sum_{i=1}^{l-1} (\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\delta}_{t+1+i}\otimes\tilde{\mathbf{x}}^s_{t+i}-\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^s_{t+i}\right)$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\delta}_{t+1}\otimes\tilde{\mathbf{x}}^f_t\otimes\tilde{\mathbf{x}}^f_t-\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)$$
$$+\sum_{i=1}^{l-1} (\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\delta}_{t+1+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}-\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)]$$

$$=\sum_{i=0}^{l-1} (\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i} \left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)$$
$$+\sum_{i=0}^{l-1} (\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i} \left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\right)$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\delta}_{t+1}$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\delta}_{t+1}\otimes\tilde{\mathbf{x}}^s_t-\mathbf{0}\right)$$
$$+\sum_{i=1}^{l-1} (\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\mathbf{0}-\mathbf{0}\right)$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\delta}_{t+1}\otimes\tilde{\mathbf{x}}^f_t\otimes\tilde{\mathbf{x}}^f_t-\mathbf{0}\right)$$
$$+\sum_{i=2}^{l-1} (\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{0}-\mathbf{0}\right)$$

because $\mathbf{x}^s_{t+i}$ is a function of $\mathbf{x}^f_{t+i}$ which is a function of $\boldsymbol{\epsilon}_{t+i}$. The zero-mean iid innovations therefore implies that, $E_t\left[\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^s_{t+i}\right)\right]=\mathbf{0}$ and $E_t\left[\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\tilde{\mathbf{x}}^s_{t+i}\right)\right]=\mathbf{0}$
The same argument implies that $E_t\left[\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)\right]=\mathbf{0}$
and $E_t\left[\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\right)\right]=\mathbf{0}$

$$=\sum_{i=0}^{l-1} (\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i} \left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)$$
$$+\sum_{i=0}^{l-1} (\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i} \left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\right)$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\nu}$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\nu}\otimes\mathbf{x}^s_t\right)$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\nu}\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)$$

the shock hitting in period $t+1\Longrightarrow\tilde{\mathbf{x}}^f_t=\mathbf{x}^f_t$ and $\tilde{\mathbf{x}}^s_t=\mathbf{x}^s_t$

$$=\sum_{i=0}^{l-1} (\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i} \left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)$$
$$+\sum_{i=0}^{l-1} (\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i} \left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\right)$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)(\boldsymbol{\nu}\otimes 1)$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\nu}\otimes\mathbf{x}^s_t\right)$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\nu}\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)$$

$$
\begin{aligned}
&= \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f\right) \\
&+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f\right) \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \mathbf{h_x}\mathbf{x}_t^s\right) \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f\right) \\
&+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f\right) \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \sigma\boldsymbol{\eta\nu} \otimes \mathbf{h_x}\mathbf{x}_t^s + \sigma\boldsymbol{\eta\nu} \otimes \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f\right) \\
&+ \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f\right) \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x}\mathbf{x}_t^s + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)
\end{aligned}
$$

as the shock hittings in period $t+1$

To derive a recursive version for this sum, we let

$$
\begin{aligned}
X_l &= \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f\right) \\
&+ \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f\right) \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x}\mathbf{x}_t^s + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)
\end{aligned}
$$

so

$$
X_1 = \sigma\boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x}\mathbf{x}_t^s + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)
$$

$$
\begin{aligned}
X_2 &= \sum_{i=1}^{1} (\mathbf{h_x} \otimes \mathbf{h_x})^{1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f\right) \\
&+ \sum_{i=1}^{1} (\mathbf{h_x} \otimes \mathbf{h_x})^{1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f\right) \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\sigma\boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x}\mathbf{x}_t^s + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}_{t+1}^f \otimes \tilde{\mathbf{x}}_{t+1}^f \otimes \tilde{\mathbf{x}}_{t+1}^f - \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) \\
&+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}_{t+1}^f - \mathbf{x}_{t+1}^f\right) \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\sigma\boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x}\mathbf{x}_t^s + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= (\mathbf{h_x} \otimes \mathbf{h_x}) X_1 + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}_{t+1}^f \otimes \tilde{\mathbf{x}}_{t+1}^f \otimes \tilde{\mathbf{x}}_{t+1}^f - \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) \\
&+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}_{t+1}^f - \mathbf{x}_{t+1}^f\right)
\end{aligned}
$$

$$
\begin{aligned}
X_3 &= \sum_{i=1}^{2} (\mathbf{h_x} \otimes \mathbf{h_x})^{2-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right) \\
&+ \sum_{i=1}^{2} (\mathbf{h_x} \otimes \mathbf{h_x})^{2-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i}\right) \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^2 \left(\sigma\boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x}\mathbf{x}^s_t + \tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}^f_{t+1} \otimes \tilde{\mathbf{x}}^f_{t+1} \otimes \tilde{\mathbf{x}}^f_{t+1} - \mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1}\right) \\
&+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}^f_{t+2} \otimes \tilde{\mathbf{x}}^f_{t+2} \otimes \tilde{\mathbf{x}}^f_{t+2} - \mathbf{x}^f_{t+2} \otimes \mathbf{x}^f_{t+2} \otimes \mathbf{x}^f_{t+2}\right) \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}^f_{t+1} - \mathbf{x}^f_{t+1}\right) \\
&+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}^f_{t+2} - \mathbf{x}^f_{t+2}\right) \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^2 \left(\sigma\boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x}\mathbf{x}^s_t + \tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= (\mathbf{h_x} \otimes \mathbf{h_x}) X_2 + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}^f_{t+2} \otimes \tilde{\mathbf{x}}^f_{t+2} \otimes \tilde{\mathbf{x}}^f_{t+2} - \mathbf{x}^f_{t+2} \otimes \mathbf{x}^f_{t+2} \otimes \mathbf{x}^f_{t+2}\right) \\
&+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}^f_{t+2} - \mathbf{x}^f_{t+2}\right)
\end{aligned}
$$

Hence, in general

$$
\begin{aligned}
X_k &= (\mathbf{h_x} \otimes \mathbf{h_x}) X_{k-1} + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}^f_{t+k-1} \otimes \tilde{\mathbf{x}}^f_{t+k-1} \otimes \tilde{\mathbf{x}}^f_{t+k-1} - \mathbf{x}^f_{t+k-1} \otimes \mathbf{x}^f_{t+k-1} \otimes \mathbf{x}^f_{t+k-1}\right) \\
&+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}^f_{t+k-1} - \mathbf{x}^f_{t+k-1}\right)
\end{aligned}
$$

11.3.3 Summarizing

At third order, the total effect on the state variables is:

$$
E_t\left[\tilde{\mathbf{x}}_{t+l} - \mathbf{x}_{t+l}\right] = E_t\left[\tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l}\right] + E_t\left[\tilde{\mathbf{x}}^s_{t+l} - \mathbf{x}^s_{t+l}\right] + E_t\left[\tilde{\mathbf{x}}^{rd}_{t+l} - \mathbf{x}^{rd}_{t+l}\right]
$$

For the control variables:

$$
\begin{aligned}
\mathbf{y}^{rd}_{t+l} &= \mathbf{g_x}\left(\mathbf{x}^f_{t+l} + \mathbf{x}^s_{t+l} + \mathbf{x}^{rd}_{t+l}\right) + \tfrac{1}{2}\mathbf{G_{xx}}\left(\left(\mathbf{x}^f_{t+l} \otimes \mathbf{x}^f_{t+l}\right) + 2\left(\mathbf{x}^f_{t+l} \otimes \mathbf{x}^s_{t+l}\right)\right) \\
&+ \tfrac{1}{6}\mathbf{G_{xxx}}\left(\mathbf{x}^f_{t+l} \otimes \mathbf{x}^f_{t+l} \otimes \mathbf{x}^f_{t+l}\right) + \tfrac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_{t+l} + \tfrac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3 \\
\tilde{\mathbf{y}}^{rd}_{t+l} &= \mathbf{g_x}\left(\tilde{\mathbf{x}}^f_{t+l} + \tilde{\mathbf{x}}^s_{t+l} + \tilde{\mathbf{x}}^{rd}_{t+l}\right) + \tfrac{1}{2}\mathbf{G_{xx}}\left(\left(\tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^f_{t+l}\right) + 2\left(\tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^s_{t+l}\right)\right) \\
&+ \tfrac{1}{6}\mathbf{G_{xxx}}\left(\tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^f_{t+l}\right) + \tfrac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2\tilde{\mathbf{x}}^f_{t+l} + \tfrac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3
\end{aligned}
$$

So:

$$
\begin{aligned}
&E_t\left[\tilde{\mathbf{y}}^{rd}_{t+l} - \mathbf{y}^{rd}_{t+l}\right] \\
&= \mathbf{g_x}\left(E_t\left[\tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l}\right] + E_t\left[\tilde{\mathbf{x}}^s_{t+l} - \mathbf{x}^s_{t+l}\right] + E_t\left[\tilde{\mathbf{x}}^{rd}_{t+l} - \mathbf{x}^{rd}_{t+l}\right]\right) \\
&+ \tfrac{1}{2}\mathbf{G_{xx}}\left(E_t\left[\tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l} \otimes \mathbf{x}^f_{t+l}\right] + 2E_t\left[\tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^s_{t+l} - \mathbf{x}^f_{t+l} \otimes \mathbf{x}^s_{t+l}\right]\right) \\
&+ \tfrac{1}{6}\mathbf{G_{xxx}}E_t\left[\tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l} \otimes \mathbf{x}^f_{t+l} \otimes \mathbf{x}^f_{t+l}\right] + \tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2 E_t\left[\tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l}\right]
\end{aligned}
$$

12 Alternative notation with σ in the state vector

When deriving the perturbation approximation, the perturbation parameter σ is treated as a variable. It may therefore be natural to consider σ as a part of the state vector when constructing the state space system for the approximated model. We therefore define $\tilde{\mathbf{x}}_t^f = \begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \sigma \end{bmatrix}'$, $\tilde{\mathbf{x}}_t^s = \begin{bmatrix} (\mathbf{x}_t^s)' & \sigma \end{bmatrix}'$, and $\tilde{\mathbf{x}}_t^{rd} = \begin{bmatrix} \left(\mathbf{x}_t^{rd}\right)' & \sigma \end{bmatrix}'$.

At first-order:

$$\mathbf{y}_t^f = \begin{bmatrix} \mathbf{g}_{\mathbf{x}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_t^f \\ \sigma \end{bmatrix}$$

$\Updownarrow$

$$\mathbf{y}_t^f = \tilde{\mathbf{g}}_{\tilde{\mathbf{x}}} \tilde{\mathbf{x}}_t^f$$

$$\begin{bmatrix} \mathbf{x}_{t+1}^f \\ \sigma \end{bmatrix} = \begin{bmatrix} \mathbf{h}_{\mathbf{x}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_t^f \\ \sigma \end{bmatrix} + \begin{bmatrix} \sigma \boldsymbol{\eta} \\ \mathbf{0} \end{bmatrix} \boldsymbol{\epsilon}_{t+1}$$

$\Updownarrow$

$$\tilde{\mathbf{x}}_{t+1}^f = \tilde{\mathbf{h}}_{\tilde{\mathbf{x}}} \tilde{\mathbf{x}}_t^f + \begin{bmatrix} \sigma \boldsymbol{\eta} \\ \mathbf{0} \end{bmatrix} \boldsymbol{\epsilon}_{t+1}$$

At second order:

$$y_t^s(i) = \begin{bmatrix} \mathbf{g}_{\mathbf{x}}(i,:) & 0 \end{bmatrix} \left(\begin{bmatrix} \mathbf{x}_t^f \\ \sigma \end{bmatrix} + \begin{bmatrix} \mathbf{x}_t^s \\ \sigma \end{bmatrix} \right) + \frac{1}{2} \begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \sigma \end{bmatrix} \begin{bmatrix} \mathbf{g}_{\mathbf{xx}}(i,:,:) & 0 \\ \mathbf{0} & g_{\sigma\sigma}(i,1) \end{bmatrix} \begin{bmatrix} \mathbf{x}_t^f \\ \sigma \end{bmatrix}$$

$\Updownarrow$

$$y_t^s(i) = \tilde{\mathbf{g}}_{\tilde{\mathbf{x}}}(i,:) \left(\tilde{\mathbf{x}}_t^f + \tilde{\mathbf{x}}_t^s \right) + \frac{1}{2} \left(\tilde{\mathbf{x}}_t^f \right)' \tilde{\mathbf{g}}_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}}(i,:,:) \tilde{\mathbf{x}}_t^f$$

$$x_t^s(i) = \begin{bmatrix} \mathbf{h}_{\mathbf{x}}(i,:) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_t^s \\ \sigma \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \sigma \end{bmatrix} \begin{bmatrix} \mathbf{h}_{\mathbf{xx}}(i,:,:) & 0 \\ \mathbf{0} & h_{\sigma\sigma}(i,1) \end{bmatrix} \begin{bmatrix} \mathbf{x}_t^f \\ \sigma \end{bmatrix}$$

$\Updownarrow$

$$x_t^s(i) = \tilde{\mathbf{h}}_{\tilde{\mathbf{x}}}(i,:) \tilde{\mathbf{x}}_t^s + \frac{1}{2} \left(\tilde{\mathbf{x}}_t^f \right)' \tilde{\mathbf{h}}_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}}(i,:,:) \tilde{\mathbf{x}}_t^f$$

for $i = 1, 2, ..., n_x$.

At third order:

$$\begin{aligned} y_t^{rd}(i) = & \begin{bmatrix} \mathbf{g}_{\mathbf{x}}(i,:) & 0 \end{bmatrix} \left(\begin{bmatrix} \mathbf{x}_t^f \\ \sigma \end{bmatrix} + \begin{bmatrix} \mathbf{x}_t^s \\ \sigma \end{bmatrix} + \begin{bmatrix} \mathbf{x}_t^{rd} \\ \sigma \end{bmatrix} \right) \\ & + \frac{1}{2} \begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \sigma \end{bmatrix} \begin{bmatrix} \mathbf{g}_{\mathbf{xx}}(i,:,:) & 0 \\ \mathbf{0} & g_{\sigma\sigma} \end{bmatrix} \left(\begin{bmatrix} \mathbf{x}_t^f \\ \sigma \end{bmatrix} + 2 \begin{bmatrix} \mathbf{x}_t^s \\ \sigma \end{bmatrix} \right) \\ & + \frac{1}{6} \begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \sigma \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \sigma \end{bmatrix} \begin{bmatrix} \mathbf{g}_{\mathbf{xxx}}(i,1,:,:) & 0 \\ \mathbf{0} & 3g(i,1)_{\sigma\sigma\mathbf{x}} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t^f \\ \sigma \end{bmatrix} \\ \begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \sigma \end{bmatrix} \begin{bmatrix} \mathbf{g}_{\mathbf{xxx}}(i,2,:,:) & 0 \\ \mathbf{0} & 3g(i,2)_{\sigma\sigma\mathbf{x}} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t^f \\ \sigma \end{bmatrix} \\ \dots \\ \begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \sigma \end{bmatrix} \begin{bmatrix} \mathbf{g}_{\mathbf{xxx}}(i,n_x,:,:) & 0 \\ \mathbf{0} & 3g(i,n_x)_{\sigma\sigma\mathbf{x}} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t^f \\ \sigma \end{bmatrix} \\ \begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \sigma \end{bmatrix} \begin{bmatrix} \mathbf{0} & 0 \\ \mathbf{0} & g(i,1)_{\sigma\sigma\sigma} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t^f \\ \sigma \end{bmatrix} \end{bmatrix} \end{aligned}$$

$\Updownarrow$

$$\begin{aligned} y_t^{rd}(i) &= \tilde{\mathbf{g}}_{\tilde{\mathbf{x}}}(i,:)\left(\tilde{\mathbf{x}}_t^f+\tilde{\mathbf{x}}_t^s+\tilde{\mathbf{x}}_t^{rd}\right) \\ &+\frac{1}{2}\left(\tilde{\mathbf{x}}_t^f\right)'\begin{bmatrix}\mathbf{g}_{\mathbf{xx}}(i,:,:) & 0\\ \mathbf{0} & g_{\sigma\sigma}\end{bmatrix}\left(\tilde{\mathbf{x}}_t^f+2\tilde{\mathbf{x}}_t^s\right) \\ &+\frac{1}{6}\left(\tilde{\mathbf{x}}_t^f\right)'\begin{bmatrix}\left(\tilde{\mathbf{x}}_t^f\right)'\begin{bmatrix}\mathbf{g}_{\mathbf{xxx}}(i,1,:,:) & 0\\ \mathbf{0} & 3g(i,1)_{\sigma\sigma\mathbf{x}}\end{bmatrix}\tilde{\mathbf{x}}_t^f\\ \left(\tilde{\mathbf{x}}_t^f\right)'\begin{bmatrix}\mathbf{g}_{\mathbf{xxx}}(i,2,:,:) & 0\\ \mathbf{0} & 3g(i,2)_{\sigma\sigma\mathbf{x}}\end{bmatrix}\tilde{\mathbf{x}}_t^f\\ \cdots\\ \left(\tilde{\mathbf{x}}_t^f\right)'\begin{bmatrix}\mathbf{g}_{\mathbf{xxx}}(i,n_x,:,:) & 0\\ \mathbf{0} & 3g(i,n_x)_{\sigma\sigma\mathbf{x}}\end{bmatrix}\tilde{\mathbf{x}}_t^f\\ \left(\tilde{\mathbf{x}}_t^f\right)'\begin{bmatrix}\mathbf{0} & 0\\ \mathbf{0} & g(i,1)_{\sigma\sigma\sigma}\end{bmatrix}\tilde{\mathbf{x}}_t^f\end{bmatrix} \end{aligned}$$

Notice that

$$\begin{bmatrix}\left(\mathbf{x}_t^f\right)' & \sigma\end{bmatrix}\begin{bmatrix}\begin{bmatrix}\left(\mathbf{x}_t^f\right)' & \sigma\end{bmatrix}\begin{bmatrix}\mathbf{g}_{\mathbf{xxx}}(i,1,:,:) & 0\\ \mathbf{0} & 3g(i,1)_{\sigma\sigma\mathbf{x}}\end{bmatrix}\begin{bmatrix}\mathbf{x}_t^f\\ \sigma\end{bmatrix}\\ \begin{bmatrix}\left(\mathbf{x}_t^f\right)' & \sigma\end{bmatrix}\begin{bmatrix}\mathbf{g}_{\mathbf{xxx}}(i,2,:,:) & 0\\ \mathbf{0} & 3g(i,2)_{\sigma\sigma\mathbf{x}}\end{bmatrix}\begin{bmatrix}\mathbf{x}_t^f\\ \sigma\end{bmatrix}\\ \cdots\\ \begin{bmatrix}\left(\mathbf{x}_t^f\right)' & \sigma\end{bmatrix}\begin{bmatrix}\mathbf{g}_{\mathbf{xxx}}(i,n_x,:,:) & 0\\ \mathbf{0} & 3g(i,n_x)_{\sigma\sigma\mathbf{x}}\end{bmatrix}\begin{bmatrix}\mathbf{x}_t^f\\ \sigma\end{bmatrix}\\ \begin{bmatrix}\left(\mathbf{x}_t^f\right)' & \sigma\end{bmatrix}\begin{bmatrix}\mathbf{0} & 0\\ \mathbf{0} & g(i,1)_{\sigma\sigma\sigma}\end{bmatrix}\begin{bmatrix}\mathbf{x}_t^f\\ \sigma\end{bmatrix}\end{bmatrix}$$

$$=\begin{bmatrix}\left(\mathbf{x}_t^f\right)' & \sigma\end{bmatrix}\begin{bmatrix}\left(\mathbf{x}_t^f\right)'\mathbf{g}_{\mathbf{xxx}}(i,1,:,:)\,\mathbf{x}_t^f+3g(i,1)_{\sigma\sigma\mathbf{x}}\,\sigma^2\\ \left(\mathbf{x}_t^f\right)'\mathbf{g}_{\mathbf{xxx}}(i,2,:,:)\,\mathbf{x}_t^f+3g(i,2)_{\sigma\sigma\mathbf{x}}\,\sigma^2\\ \cdots\\ \left(\mathbf{x}_t^f\right)'\mathbf{g}_{\mathbf{xxx}}(i,n_x,:,:)\,\mathbf{x}_t^f+3g(i,n_x)_{\sigma\sigma\mathbf{x}}\,\sigma^2\\ g(i,1)_{\sigma\sigma\sigma}\,\sigma^2\end{bmatrix}$$

$$=\sum_{k=1}^{n_x} x_t^f(k)\left(\left(\mathbf{x}_t^f\right)'\mathbf{g}_{\mathbf{xxx}}(i,k,:,:)\,\mathbf{x}_t^f+3g(i,k)_{\sigma\sigma\mathbf{x}}\,\sigma^2\right)+g(i,1)_{\sigma\sigma\sigma}\,\sigma^3$$

as desired.

$$
\begin{aligned}
x_t^{rd}(i) &= \left[\begin{array}{cc} \mathbf{h_x}(i,:) & 0 \end{array}\right]\left[\begin{array}{c} \mathbf{x}_t^{rd} \\ \sigma \end{array}\right] \\
&+\left[\begin{array}{cc} \left(\mathbf{x}_t^f\right)' & \sigma \end{array}\right]\left[\begin{array}{cc} \mathbf{h_{xx}}(i,:,:) & 0 \\ \mathbf{0} & h_{\sigma\sigma} \end{array}\right]\left[\begin{array}{c} \mathbf{x}_t^s \\ \sigma \end{array}\right] \\
&+\frac{1}{6}\left[\begin{array}{cc} \left(\mathbf{x}_t^f\right)' & \sigma \end{array}\right]\left[\begin{array}{c}
\left[\begin{array}{cc} \left(\mathbf{x}_t^f\right)' & \sigma \end{array}\right]\left[\begin{array}{cc} \mathbf{h_{xxx}}(i,1,:,:) & 0 \\ \mathbf{0} & 3h(i,1)_{\sigma\sigma\mathbf{x}} \end{array}\right]\left[\begin{array}{c} \mathbf{x}_t^f \\ \sigma \end{array}\right] \\
\left[\begin{array}{cc} \left(\mathbf{x}_t^f\right)' & \sigma \end{array}\right]\left[\begin{array}{cc} \mathbf{h_{xxx}}(i,2,:,:) & 0 \\ \mathbf{0} & 3h(i,2)_{\sigma\sigma\mathbf{x}} \end{array}\right]\left[\begin{array}{c} \mathbf{x}_t^f \\ \sigma \end{array}\right] \\
\dots \\
\left[\begin{array}{cc} \left(\mathbf{x}_t^f\right)' & \sigma \end{array}\right]\left[\begin{array}{cc} \mathbf{h_{xxx}}(i,n_x,:,:) & 0 \\ \mathbf{0} & 3h(i,n_x)_{\sigma\sigma\mathbf{x}} \end{array}\right]\left[\begin{array}{c} \mathbf{x}_t^f \\ \sigma \end{array}\right] \\
\left[\begin{array}{cc} \left(\mathbf{x}_t^f\right)' & \sigma \end{array}\right]\left[\begin{array}{cc} \mathbf{0} & 0 \\ \mathbf{0} & h(i,1)_{\sigma\sigma\sigma} \end{array}\right]\left[\begin{array}{c} \mathbf{x}_t^f \\ \sigma \end{array}\right]
\end{array}\right]
\end{aligned}
$$

$\Updownarrow$

$$
\begin{aligned}
x_t^{rd}(i) &= \left[\begin{array}{cc} \mathbf{h_x}(i,:) & 0 \end{array}\right]\tilde{\mathbf{x}}_t^{rd} \\
&+\left(\tilde{\mathbf{x}}_t^f\right)'\left[\begin{array}{cc} \mathbf{h_{xx}}(i,:,:) & 0 \\ \mathbf{0} & h_{\sigma\sigma} \end{array}\right]\tilde{\mathbf{x}}_t^f \\
&+\frac{1}{6}\left(\tilde{\mathbf{x}}_t^f\right)'\left[\begin{array}{c}
\left(\tilde{\mathbf{x}}_t^f\right)'\left[\begin{array}{cc} \mathbf{h_{xxx}}(i,1,:,:) & 0 \\ \mathbf{0} & 3h(i,1)_{\sigma\sigma\mathbf{x}} \end{array}\right]\tilde{\mathbf{x}}_t^f \\
\left(\tilde{\mathbf{x}}_t^f\right)'\left[\begin{array}{cc} \mathbf{h_{xxx}}(i,2,:,:) & 0 \\ \mathbf{0} & 3h(i,2)_{\sigma\sigma\mathbf{x}} \end{array}\right]\tilde{\mathbf{x}}_t^f \\
\dots \\
\left(\tilde{\mathbf{x}}_t^f\right)'\left[\begin{array}{cc} \mathbf{h_{xxx}}(i,n_x,:,:) & 0 \\ \mathbf{0} & 3h(i,n_x)_{\sigma\sigma\mathbf{x}} \end{array}\right]\tilde{\mathbf{x}}_t^f \\
\left(\tilde{\mathbf{x}}_t^f\right)'\left[\begin{array}{cc} \mathbf{0} & 0 \\ \mathbf{0} & h(i,1)_{\sigma\sigma\sigma} \end{array}\right]\tilde{\mathbf{x}}_t^f
\end{array}\right]
\end{aligned}
$$

References

Schmitt-Grohé, S. & Uribe, M. (2004), ‘Solving dynamic general equilibrium models using a second-order approximation to the policy function’, *Journal of Economic Dynamics and Control* **28**, 755–775.