Online Appendix
Idea Flows, Economic Growth, and Trade

Fernando Alvarez, Francisco J. Buera, and Robert E. Lucas, Jr.

November 9, 2013

The results in this appendix are expressed in terms of cost, i.e., the reciprocal of productivity. It is straightforward to express them in terms of productivity as we do in the main body of the paper.

D Interpretation of θ

In Section 2 we show that there are two simple statistics that determine when an initial distribution $F(z,0)$ belong to the basin of attraction of a balance growth: the curvature θ and scale λ parameters. These two parameters, together with arrival rate of meetings, α, fully characterize the growth rate and the stationary distribution of cost in a balance growth path. Of the two parameters characterizing the initial distribution, θ is asymptotically more important as it governs the growth rate. This appendix relates the curvature parameter with the elasticity of CDF at zero, and provide an interpretation for this parameter.

The following result relates θ to the elasticity at 0 of the initial cdf $1-F(z,0)$.

Lemma D.1. Suppose that the density of the right cdf $F(z,0)$ satisfies property (11) for some $\theta > 0$ and $\lambda > 0$. Then

$$\lim_{z\to 0} \frac{f(z,0)z}{1-F(z,0)} = \frac{1}{\theta}\ .$$

Proof: Rearranging equation (11) we obtain

$$\begin{aligned}
\lambda &= \lim_{z\to 0} \theta \frac{1-F(z^\theta,0)}{z} \frac{f(z^\theta,0)z^\theta}{1-F(z^\theta,0)} \\
&= \theta \lim_{z\to 0} \frac{1-F(z^\theta,0)}{z} \lim_{z\to 0} \frac{f(z^\theta,0)z^\theta}{1-F(z^\theta,0)} \\
&= \theta \lim_{z\to 0} \theta f(z^\theta,0) z^{\theta-1} \lim_{z\to 0} \frac{f(z^\theta,0)z^\theta}{1-F(z^\theta,0)} \\
&= \theta\lambda \lim_{z\to 0} \frac{f(z^\theta,0)z^\theta}{1-F(z^\theta,0)}.
\end{aligned}$$

The second but last equality follows from applying l'Hospital's rule to the first limit, and the last equality follows from condition (11). Thus, the desired result follows from the last line. □

The following Lemma provides an interpretation of θ as a measure of the relative concentration of arbitrarily low costs in two distributions.

Lemma D.2: Let $1-F_1(z)$ and $1-F_2(z)$ be two cdf with continuous and strictly positive elasticity at 0, i.e., $\lim_{z\to 0} \epsilon_i(z) = \epsilon_i(0) > 0$, and $\epsilon_1(0) = 1/\theta_1 < \epsilon_2(0) = 1/\theta_2$ ($\theta_1 > \theta_2$). Then

$$\lim_{z\to 0} \frac{1-F_2(z)}{1-F_1(z)} = \lim_{z\to 0} \frac{f_2(z)}{f_1(z)} = 0.$$

Proof: Using the definition of the elasticity we can write the cdf and density functions as

$$1 - F_i(z) = [1 - F_i(\bar{z})] \exp\left[-\int_z^{\bar{z}} \frac{\epsilon_i(y)}{y} dy\right].$$

and

$$f_i(z) = [1 - F_i(\bar{z})] \frac{\epsilon_i(z)}{z} \exp\left[-\int_z^{\bar{z}} \frac{\epsilon_i(y)}{y} dy\right].$$

Therefore, we can express the ratio between the first and second density functions as

$$\frac{f_2(z)}{f_1(z)} = \frac{1 - F_2(\bar{z})}{1 - F_1(z)} \frac{\epsilon_2(z)}{\epsilon_1(z)} \exp\left[-\int_z^{\bar{z}} \frac{\epsilon_2(y) - \epsilon_1(y)}{y} dy\right].$$

From the continuity of $\epsilon_i(z)$ and the fact that $\epsilon_2(0) - \epsilon_1(0) > 0$ we know that for $\bar{z}$ close to zero there exist ε, $0 < \varepsilon \leq \epsilon_2(0) - \epsilon_1(0)$, such that $\epsilon_2(z) - \epsilon_1(z) \geq \varepsilon$ for all $0 < z \leq \bar{z}$. Therefore, for all $z \leq \bar{z}$

$$\begin{aligned} \frac{f_2(z)}{f_1(z)} &\leq \frac{1 - F_2(\bar{z})}{1 - F_1(z)} \frac{\epsilon_2(z)}{\epsilon_1(z)} \exp\left[-\int_z^{\bar{z}} \frac{\varepsilon}{y} dy\right] \\ &= \frac{1 - F_2(\bar{z})}{1 - F_1(z)} \frac{\epsilon_2(z)}{\epsilon_1(z)} \left(\frac{z}{\bar{z}}\right)^{\varepsilon}, \end{aligned}$$

and since $f_i(z) \geq 0$, $i = 1, 2$,

$$\lim_{z \to 0} \frac{f_2(z)}{f_1(z)} = 0.$$

By l'Hopital rule, this also implies that

$$\lim_{z \to 0} \frac{1 - F_2(z)}{1 - F_1(z)} = 0.$$

□

E Example of Non-Convergent Initial Distributions

There are, of course, initial distributions that generate paths that do not converge in the sense define in Section 2: any distribution with a support that is bounded away from 0, for example. A log normal $F(., 0)$ has an elasticity of $1 - F(., 0)$ that converges to ∞ and so Lemma D.1 implies $\theta = 0$. In this case, the economy does not have a balanced growth path with strictly positive growth. In the opposite extreme, an example of a distribution with an elasticity converging to zero is

$$1 - F(z, 0) = \exp\left[-\sum_{i=1}^{\infty} \left(\frac{\beta}{\delta}\right)^i \left(1 - z^{\delta^i}\right)\right], \; z \in [0, 1], \; 0 < \delta < \beta < 1.$$

The elasticity of $1 - F(z,0)$ equals $\sum_{i=1}^{\infty} \beta^i z^{\delta^i}$, and therefore, it tends to 0 as $z \to 0$. In this case the economy does not have a balanced growth path since the growth rate will be increasing without bound as time passes.

Initial distributions with a strictly positive elasticity at zero but that fail to satisfy (11) can also be constructed. One example is

$$1 - F(z,0) = z \exp\left[-\sum_{i=1}^{\infty}\left(\frac{\beta}{\delta}\right)^i \left(1 - z^{\delta^i}\right)\right], \; z \in [0,1], \; 0 < \delta < \beta < 1.$$

The elasticity of $1-F(z,0)$ equals $1+\sum_{i=1}^{\infty} \beta^i z^{\delta^i}$, and therefore tends to 1 as $z \to 0$, but this cdf does satisfy condition (11). In this case, $\lim_{z\to 0}(1-F(z^\theta,0))/z = 0\ (\infty)$ for all $\theta \leq (>)1$, which implies that condition (11) is not satisfied for any θ.

F Partial Converse to Lemma D.1

Can the conditions for an initial distribution to belong to the basin of attraction of a balance growth path be express solely in terms of the behavior of the elasticity around zero? We restrict the set of initial conditions so that if they have a bounded elasticity then they satisfy (11). Consider the class of initial c.d.f. whose elasticity on the neighborhood of zero can be written as a sum of power functions, i.e.,

$$\epsilon(z) \equiv \frac{f(z,0)z}{1-F(z,0)} = e_0 + \sum_{i=1}^{\infty} e_i \; z^{\xi_i} + o(z) \tag{F.1}$$

where $\xi_i > 0$ and $\lim_{z\downarrow 0} \frac{o(z)}{z} \to 0$.[1]

The follow result provides a partial converse to Proposition 3, as it provides a sufficient condition for a initial distribution with strictly positive and finite elasticity to converges to a balance growth path as defined in (10).

Proposition F.1 Assume that the initial c.d.f. $F(z,0)$ has elasticity of the form given by (F.1), with $\sum_{i=1}^{\infty} |\frac{e_i}{\xi_i}| = A < \infty$. Then, $F(z,0)$ satisfies condition (11).

Proof. Define $H(x) = F(x^\theta, 0)$, where $\theta = 1/e_0$. Let $0 \leq x < \bar{x} < 1$ closed enough to 0.

[1]This class includes the cases in which $\epsilon(z)$ is differentiable at 0 (set $\xi_i \geq 1$ for all i), as well as many other cases where it is not differentiable, such as $\epsilon(z) = e_0 + \sqrt{z}$.

Integrating the equation defining the elasticity of $H(x)$ between x and $\bar{x}$ we obtain

$$\begin{aligned} H(x) &= H(\bar{x}) \exp\left[-\int_x^{\bar{x}} \frac{\epsilon(t)/\epsilon(0)}{t} dt\right] \\ &= H(\bar{x}) \exp\left[-\int_x^{\bar{x}} \frac{1+\frac{1}{\epsilon(0)}\left[\sum_{i=1}^{\infty} e_i\, t^{\xi_i} + o(t)\right]}{t} dt\right] \\ &= H(\bar{x}) \frac{x}{\bar{x}} \exp\left[-\frac{1}{\epsilon(0)} \int_x^{\bar{x}} \sum_{i=1}^{\infty} e_i\, t^{\xi_i - 1} dt - \frac{1}{\epsilon(0)} \int_x^{\bar{x}} \frac{o(t)}{t} dt\right]. \end{aligned}$$

where the second equality follows from the definition of a regular elasticity. Rearrenging,

$$\begin{aligned} \frac{H(x)}{x} = & \quad \frac{H(\bar{x})}{\bar{x}} \exp\left[-\frac{1}{\epsilon(0)} \int_x^{\bar{x}} \sum_{i=1}^{\infty} e_i\, t^{\xi_i-1} dt\right] \exp\left[-\frac{1}{\epsilon(0)} \int_x^{\bar{x}} \frac{o(t)}{t} dt\right] \\ = & \quad \frac{H(\bar{x})}{\bar{x}} \exp\left[-\frac{1}{\epsilon(0)} \left[\int_x^{\bar{x}} \sum_{i \in pos} e_i\, t^{\xi_i-1} dt - \int_x^{\bar{x}} \sum_{i \in neg} |e_i|\, t^{\xi_i-1} dt\right]\right] \\ & \quad \exp\left[-\frac{1}{\epsilon(0)} \int_x^{\bar{x}} \frac{o(t)}{t} dt\right] \end{aligned} \tag{F.2}$$

where $pos = \{i : e_i \geq 0\}$ and $neg = \{i : e_i < 0\}$. We notice that each of the integrals inside the first exponential increase when we lower x, as we are integrating a positive function over a larger range. Moreover, these two integrals are bounded, i.e.,

$$\begin{aligned} \int_x^{\bar{x}} \sum_{i \in pos} e_i\, t^{\xi_i-1} dt &\leq \int_x^{\bar{x}} \sum_{i=1}^{\infty} |e_i|\, t^{\xi_i-1} dt \\ &\leq \liminf_{n\to\infty} \int_x^{\bar{x}} \sum_{i=1}^{n} |e_i|\, t^{\xi_i-1} dt \\ &= \liminf_{n\to\infty} \sum_{i=1}^{n} |\frac{e_i}{\xi_i}| \left[\bar{x}^{\xi_i} - x^{\xi_i}\right] \\ &\leq \sum_{i=1}^{\infty} |\frac{e_i}{\xi_i}| = A < \infty \end{aligned}$$

where the first inequality follows from Fatou's Lemma, the second equality follows from integration, and the last inequality follows from the definition of a regular elasticity and the fact that $0 < x < \bar{x} \leq 1$. Similar arguments can be applid to show that the second integral inside of the first exponential is bounded. Finally, the absolute value of the argument of the second exponential is uniformly bounded for $\bar{x}$ small enough, i.e., $|\int_0^{\bar{x}} \frac{o(t)}{t} dt| \leq B < \infty$. Since all the integrals in the right hand side of (F.2) are monotone and bounded, they converge to a finite limit. This proves that $0 < \lim_{x\to0} H'(x) = \lim_{x\to0} H(x)/x = \lim_{x\to0} \theta x^{\theta-1} f(x^\theta, 0) < \infty$, which is equivalent to (11) as Lemma D.1 shows. $\square$

G Alternative Definition of a Balance Growth Path

For completeness, we characterize the asymptotic behavior of initial distributions that do not satisfy the conditions of Proposition F.1. While these initial distributions do not converge to a balance growth path as define in (10), they converge to a balance growth path in a weaker sense as stated in the following proposition.

Proposition G.1 Assume that the initial cdf $1 - F(z, 0)$ has an elasticity that is continuous, strictly positive, and finite at zero equal $1/\theta$. Let the $q(t)$ be the q^{th} quantile of the distribution $F(z, t)$, i.e.,

$$F(d(t), t) = \exp\left[\log\left[F\left(q(t), 0\right)\right] e^{\alpha t}\right] = 1 - q. \tag{G.1}$$

Then, the distribution of cost normalized by the q^{th} quantile converges to a Weibull with parameters θ and $\lambda = e^{1-q}$, i.e.,

$$\lim_{t\to\infty} F(z\ q(t), t) = \exp\left(-\lambda z^{1/\theta}\right). \tag{G.2}$$

and the q^{th} quantile of the cost distribution decreases at an asymptotically constant rate α/θ, i.e.,

$$\lim_{t\to\infty} \frac{1}{q(t)} \frac{\partial q(t)}{\partial t} = -\frac{\alpha}{\theta}. \tag{G.3}$$

The proof of this Proposition uses the following Lemma.

Lemma G.2: If the cdf $1 - F(z, 0)$ has an elasticity that is continuous, strictly positive, and finite at zero equals to $\epsilon(0)$, then

$$\lim_{z\to 0} \frac{F'(kx, 0)}{F'(x, 0)} = k^{\epsilon(0)-1}, \text{ for all } k > 0.$$

Proof: Using the definition of the elasticity and letting $k < 1$ (a similar argument applies for the case $k > 1$)

$$\frac{F'(kz, 0)}{F'(z, 0)} = \frac{1}{k} \frac{\epsilon(kz)}{\epsilon(z)} e^{-\int_{kz}^{z} \frac{\epsilon(u)}{u} du}.$$

From the continuity of the elasticity we know that for every ς there exist a z such that $\epsilon(0) - \varsigma \leq \epsilon(u) \leq \epsilon(0) + \varsigma$ for all $u < z$. Therefore,

$$k^{\epsilon(0)+\varsigma-1} \leq \frac{F'(kz, 0)}{F'(z, 0)} \leq k^{\epsilon(0)-\varsigma-1}.$$

Since ς can be made arbitrarily small, we obtain the desired result. $\square$

Proof of Proposition G.1. Taking the limit as $t \to \infty$ in both sides of equation (G.2)

$$\begin{aligned}
\lim_{t\to\infty} F(z\ q(t),t) &= \lim_{t\to\infty} \exp\left[\log(F(z\ q(t),0))e^{\alpha t}\right] \\
&= \exp\left[\lim_{t\to\infty} \frac{F'(z\ q(t),0)z\ q'(t)}{-\alpha e^{-\alpha t}}\right] \\
&= \exp\left[\lim_{t\to\infty} -e^{1-q}\frac{F'(z\ q(t),0)}{F'(q(t),0)}z\right] \\
&= \exp\left(-\lambda z^{1/\theta}\right),
\end{aligned}$$

where the third equality uses that $q'(t) = \alpha \log\left[F(q(t)\right] F(q(t),0)/F'(q(t),0)$, which itself follows from applying the implicit function theorem to equation (G.1), and the last equality follows from the following Lemma. Finally, we derive equation (G.3)

$$\begin{aligned}
\lim_{t\to\infty} \frac{\frac{\partial q(t)}{\partial t}}{q(t)} &= \lim_{t\to\infty} \frac{\frac{\alpha \log(F(q(t),0)F(q(t),0)}{F'(q(t),0)}}{q(t)} \\
&= -\alpha \lim_{t\to\infty} \frac{log(F(q(t),0)}{1-F(q(t),0)}\left(\frac{1-F(q(t),0)}{-F'(q(t),0)q(t)}\right) \\
&= -\frac{\alpha}{\theta},
\end{aligned}$$

where the first equality follows from applying the Implicit Function Theorem to equation (G.1), and the last equality following from L'Hopital rule to the first term and the fact that the elasticity at zero of the c.d.f. equals $1/\theta$. □

This result describes the asymptotic behavior of economies whose initial distribution of cost has a strictly positive and finite elasticity at zero but do not satisfy the condition in Proposition F.1, e.g., the distribution in last example described in Section E. Proposition G.1 shows that in these economies costs decrease asymptotically at a constant rate α/θ. Nevertheless, in these economies the distribution of costs normalized by a constant growth factor, $e^{\frac{\alpha}{\theta}t}$, is asymptotically degenerate. What happens in this example is that along most of the transition costs decrease a rate that is bounded away from α/θ.

Notice that Proposition G.1 is very related to results in the mathematical statistic literature on extreme distributions. In particular for the maximum of an infinite sequence of iid variables with finite upper bound. In that case the conditions on the elasticity is essentially the same as the von Mises condition, and the invariant distribution is Weibull. See, for example, Theorem 3.3.12 of Embrechts et al. (2003). Yet our result in Propositions 1, 2 and G.1 are different in an important way from the standard results in extreme distributions. In our set-up we obtain *geometric* growth, while, in the language of the extreme distributions, the standard result is a *linear* norming constraint, or linear growth in term of economics. Indeed the standard set-up in the mathematical statistical literature is closer to the set-up in economic models of diffusion of technologies with an exogenous idea source, such as the one by Kortum (1997). In these type of models there is no growth asymptotically. Furthermore, since in our framework there is growth asymptotically we need to focus in a stronger notion of convergence. This leads to a smaller set of initial distribution that are stable, i.e., those satisfying condition (11).

H Bertrand Competition

We consider a world economy with many locations per country, but a single producer per location, following the extension with many locations described in Appendix C. In this case perfect competition does not provide a natural benchmark. Instead we assume that producers engage in Bertrand competition: in each location the seller of each good s is still going to be the lowest cost producer, but will now charge the minimum between the cost of the second lowest cost producer and the monopolist price.

This extension has become popular in the trade literature (Bernard et al., 2003). First, it provides a simple model where the size of a firm is determined, and therefore, the theory has implications for the size distribution of firms. More recently, the literature has studied the effect of trade policy on the distribution of market and the pro-competitive gains from trade, and have stressed the importance of assumptions on the distribution of productivities to determine their magnitude (Arkolakis et al., 2012; Holmes et al., 2012).

To simplify the analysis we are going to assume that there are the same number of location in each country, i.e., $m_i = m$, and that locations are symmetric within each country, i.e., $F_{i,l}(z,t) = F_{i,l'}(z,t)$. Thus, for each good s there are $m\,n$ potential producers that have generically heterogeneous cost. Because prices in Bertrand competition depends also on the second lowest cost producer, we enlarge the profile of cost of a good s to include the cost of all locations across all countries $\mathbf{z} = (z_{1,1}, ..., z_{i,l}, ..., z_{n,m})$.

As before we let $p_i(\mathbf{z},t)$ be the price paid for good $\mathbf{z}$ in country i at t. Bertrand competition implies that

$$p_i(\mathbf{z},t) = \min_{(j,l)} \min \left\{ \frac{\eta}{\eta-1} \frac{w_j(t)}{\kappa_{ij}} z_{jl}, \min_{(j',l')\neq(j,l)} \left\{ \frac{w_{j'}(t)}{\kappa_{ij'}} z_{j'l'} \right\} \right\},$$

where the minimization is over all countries and locations pairs (j,l), and for each country pair we minimize between the monopolist price and the second lowest limit price. As was the case with perfect competition, the price index p_i must be calculated country by country.

Consumption of good $\mathbf{z}$ in country i equals

$$c_i(\mathbf{z}) = \left(\frac{p_i}{p_i(\mathbf{z})}\right)^{\eta} C_i = \left(\frac{p_i}{p_i(\mathbf{z})}\right)^{\eta} \frac{w_i L_i + \pi_i}{p_i}.$$

where π_i are the profits of firms in country i. The first equality follows from individual maximization and the second follows from the budget constraint $p_i C_i = w_i L_i + \pi_i$ since we have assumed that trade is balanced in each period.

The definition of equilibrium is standard. As in the competitive case to find an equilibrium we need to find a vector of wages $w = (w_1, ..., w_n)$ for which the implied demand for labor is equal to the inelastically given supply for each country. While the derive demand for labor is similar, it now reflects the fact that individual prices are different and profits are positive. In particular, the solution of an equilibrium boils down to find the wage vector w for which the excess derived demand equals zero, $Z(w) = 0$. As in the case with perfect competition, the price index p_i must be calculated country by country,

$$
\begin{aligned}
p_i^{1-\eta} &= \int p_i(\mathbf{z})^{1-\eta} f(\mathbf{z}) d\mathbf{z} \\
&= \sum_{(j,l)} \int_0^\infty \Bigg[\sum_{(j',l')\neq(j,l)} \int_{a_{ijj'} z_{jl}}^{a_{ijj'}\bar{m} z_{jl}} \left(\frac{z_{j'l'} w_{j'}}{\kappa_{ij'}} \right)^{1-\eta} f_{j'l'}(z_{j'l'}) \\
&\qquad \times \prod_{(j'',l'')\neq(j,l),(j',l')} F_{j''l''}\left(a_{ij'j''} z_{j'l'}\right) dz_{j'l'} \\
&\qquad + \left(\bar{m} \frac{z_{jl} w_j}{\kappa_{ij}} \right)^{1-\eta} \prod_{(j',l')\neq(j,l)} F_{j'l'}\left(\bar{m} a_{ijj'} z_{jl}\right) \Bigg] f_{jl}(z_{jl}) dz_{jl}
\end{aligned}
$$

where we first sum over the source country and location pairs (j,l) and then we integrate over the cost of the source country and location pairs z_{jl}. The first term inside of the integral considers the goods for which the price is determined by the cost of the second lowest cost source $(j',l') \neq (j,l)$, $\frac{z_{j'l'} w_{j'}}{\kappa_{ij'}} \leq \bar{m} \frac{z_{jl} w_j}{\kappa_{ij}}$, where $\bar{m}$ denotes the monopolistic markup, which equals $\frac{\eta}{\eta-1}$ if $\eta > 1$ or ∞ otherwise. The second term corresponds to the goods for which the lowest cost source can charge the monopolist price, $\bar{m} \frac{w_j z_j}{\kappa_{ij}}$.

The expression for the excess derived demand for labor of country i equals

$$
\begin{aligned}
Z_i(w) &= \sum_{l=1}^{m} \sum_{j=1}^{n} \int_0^\infty \Bigg[\sum_{(j',l')\neq(i,l)} \int_{a_{jij'}\ z_{il}}^{a_{jij'}\bar{m} z_{il}} \left(\frac{\kappa_{jj'} p_j}{w_{j'} z_{j'l'}} \right)^{\eta} f_{j'}(z_{j'l'}) \\
&\times \prod_{(j'',l'')\neq(j',l'),(i,l)} F_{j''}\left(a_{jj'j''}\ z_{j'l'}\right) dz_{j'l'} \\
&+ \left(\frac{\kappa_{ji} p_j}{\bar{m} z_{il} w_i} \right)^{\eta} \prod_{(j',l')\neq(i,l)} F_{j'}\left(\bar{m} a_{jij'} z_{il}\right) \Bigg] \frac{w_j L_j + \pi_j(w)}{p_j} \frac{z_{il}}{\kappa_{ji}} f_i(z_{il})\ dz_{il} - L_i.
\end{aligned}
$$

where, for a given w, the profits π_i are the solution of the following linear system of equations

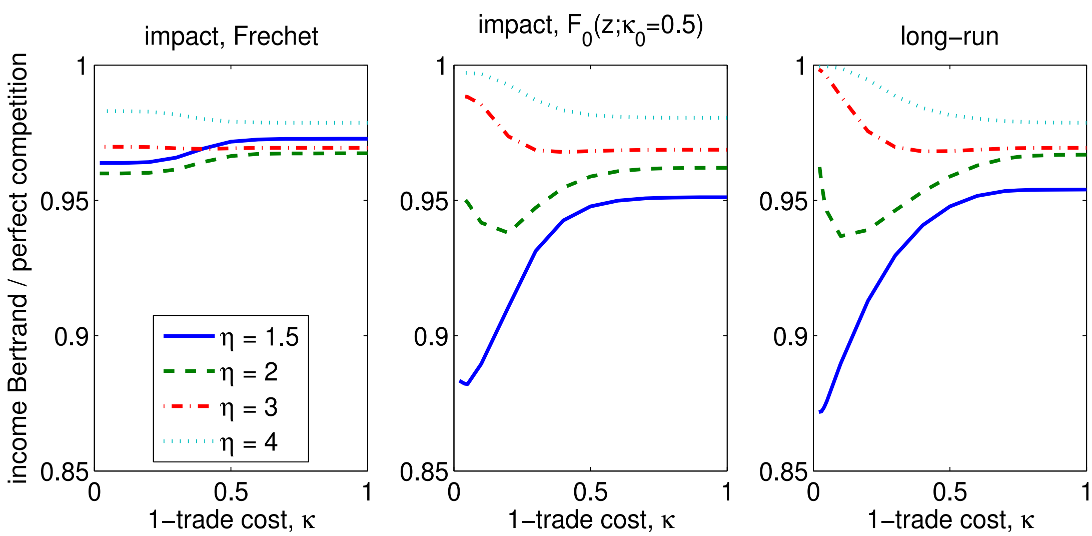
$$
\begin{aligned}
\pi_i(w) &= \sum_{l=1}^{m} \sum_{j=1}^{n} \int_0^\infty \Bigg[\sum_{(j',l')\neq(i,l)} \int_{a_{jij'}\ z_{il}}^{a_{jij'}\bar{m} z_{il}} \left(\frac{w_{j'} z_{j'l'}}{\kappa_{jj'}} - \frac{w_i z_{il}}{\kappa_{ji}} \right) \left(\frac{\kappa_{jj'} p_j}{w_{j'} z_{j'l'}} \right)^{\eta} f_{j'}(z_{j'l'}) \\
&\times \prod_{(j'',l'')\neq(j',l'),(i,l)} F_{j''}\left(a_{jj'j''}\ z_{j'l'}\right) dz_{j'l'} \\
&+ (\bar{m}-1) \frac{w_i z_{il}}{\kappa_{ji}} \left(\frac{\kappa_{ji} p_j}{\bar{m} z_{il} w_i} \right)^{\eta} \prod_{(j',l')\neq(i,l)} F_{j'}\left(\bar{m} a_{jij'} z_{il}\right) \Bigg] \frac{w_j L_j + \pi_j(w)}{p_j} f_i(z_{il})\ dz_{il}.
\end{aligned}
$$

H.1 Quantitative Exploration

We explore the robustness of the welfare results presented in Figure 2 using the Bertrand competition model described in this Appendix. We analyze a case of 25 symmetric countries, each with two locations, and set all the other parameters to be same as in the economy with perfect competition. We compare the economy with Bertrand competition with an economy with 25 symmetric countries, each with two locations, in which we assume that producers behave competitive.

In Figure H.1 we present three panels. In each panel the y-axis shows the ratio of per-capita income in the economy with Bertrand competition relative to the per-capita income in the perfectly competitive economy. In the x-axis we vary the common value of κ. We show these effects for different values of the paramete η, which controls the elasticity of substitution. The first two panels consider the impact effect, before any diffusion of technology has taken place. The last panel consider the long-run effect. The initial distribution used to calculate all the examples in the three panels has the same θ and λ. We denote by κ_0 the trade cost associated with the initial distribution. In the first panel the initial distribution is the stationary distribution for the case of costless trade, $\kappa_0 = 1$, which is Frechet. In the second panel the initial distribution is the stationary distribution of an economy with $\kappa_0 = 0.5$, which as shown in Figure 1 deviates significantly from Frechet. The value of κ_0 is irrelevant for the last panel.

Figure H.1: Comparison of the effect of trade costs in Bertrand competition relative to perfect competition.



I Poisson Arrival of Ideas

In this section we explore the implications of a version of our model with Poisson arrival of ideas. As shown in Alvarez et al. (2008), in the Poisson case the evolution of the distribution of productivity (the inverse of cost) of country i solves

$$\frac{\partial \log(F_i(z,t))}{\partial t} = -\alpha \left[1 - G_i(z,t)\right]. \tag{I.1}$$

where

$$G_i(z,t) = \sum_{j=1}^{n} \int_0^z f_j(y,t) \prod_{k\neq j} F_k\left(\frac{w_k \kappa_{ij}}{w_j \kappa_{ik}} y, t\right) dy. \qquad \text{(I.2)}$$

The following proposition characterizes the balance growth path of a symmetric world economy with costless trade.

Proposition A.5: Assume that ideas arrive with a Poisson process with intensity α from the distribution of sellers to a country. For each country $i = 1, ..., n$, $F_i(z,0) = F(z,0)$ with density satisfying

$$\lim_{z\to\infty} \frac{f(z,0)}{\frac{1}{\theta} z^{-\frac{1}{\theta}-1}} = \lambda. \qquad \text{(I.3)}$$

Then, the steady state distribution for each country is

$$\lim_{t\to\infty} F\left(e^{\nu\, t} z, t\right) = \frac{1}{\left(1 + n\lambda z^{-\frac{1}{\theta}}\right)^{1/n}}$$

and the growth rate in a balance growth path

$$\nu = n\alpha\theta.$$

Proof: Specializing equation (I.2) for the case of symmetric countries and substituting into (I.1)

$$\frac{\partial \log(F(z,t))}{\partial t} = -\alpha \left[1 - F(z,t)^n\right].$$

Multiplying and dividing the left hand side by n

$$\frac{n}{n}\frac{\partial \log(F(z,t))}{\partial t} = -\alpha \left[1 - F(z,t)^n\right]$$

$$\frac{1}{n}\frac{\partial \log(F(z,t)^n)}{\partial t} = -\alpha \left[1 - F(z,t)^n\right]$$

Defining $H(z,t) = F(z,t)^n$

$$\frac{\partial \log(H(z,t))}{\partial t} = -n\alpha \left[1 - H(z,t)\right]$$

The solution of this equation is

$$H(z,t) = \frac{1}{\frac{1-H(z,0)}{H(z,0)} e^{n\alpha t} + 1}$$

Defining the detrended productivity $x = ze^{-\nu t}$ and taking the limit as $t \to \infty$

$$\lim_{t\to\infty} H(xe^{\nu t}, t) = \lim_{t\to\infty} \frac{1}{\frac{1-H(xe^{\nu t},0)}{H(xe^{\nu t},0)} e^{n\alpha t} + 1}$$

$$= \frac{1}{\lim_{t\to\infty} \frac{1-H(xe^{\nu t},0)}{(xe^{\nu t})^{-\frac{1}{\theta}}} \left(xe^{\nu t}\right)^{-\frac{1}{\theta}} e^{n\alpha t} + 1}$$

Using assumption (I.3), l'Hopital's rule, and that $H(z,t) = F(z,t)^n$

$$\begin{aligned}
\lim_{z\to\infty} \frac{1-H(z,0)}{z^{-\frac{1}{\theta}}} &= \lim_{z\to\infty} \frac{1-F(z,0)^n}{z^{-\frac{1}{\theta}}} \\
&= \lim_{z\to\infty} \frac{nF(z,0)^{n-1}f(z,0)}{\frac{1}{\theta}z^{-\frac{1}{\theta}-1}} \\
&= n \lim_{z\to\infty} \frac{f(z,0)}{\frac{1}{\theta}z^{-\frac{1}{\theta}-1}} \\
&= n\lambda.
\end{aligned}$$

Using this we obtain

$$\lim_{t\to\infty} H(xe^{\nu t}, t) = \frac{1}{n\lambda x^{-\frac{1}{\theta}} \lim_{t\to\infty} e^{(n\alpha - \frac{\nu}{\theta})t} + 1}$$

From the last expression it follows that, in order for the right hand side not to imply a degenerate distribution, $\nu = n\alpha\theta$. In this case, the stationary distribution equals

$$\lim_{t\to\infty} H(xe^{\nu t}, t) = \frac{1}{n\lambda x^{-\frac{1}{\theta}} + 1}.$$

Finally, using that $F(z,t) = H(z,t)^{\frac{1}{n}}$ we obtain

$$\begin{aligned}
\lim_{t\to\infty} F(xe^{\nu t}, t) &= \lim_{t\to\infty} H(xe^{\nu t}, t)^{\frac{1}{n}} \\
&= \frac{1}{\left(1 + n\lambda x^{-\frac{1}{\theta}}\right)^{\frac{1}{n}}}.
\end{aligned}$$

□

It is interesting that the stationary distribution for the case of a Poisson process is a function of the number of countries n. In contrast, when ideas arrive at a deterministic rate α, the stationary distribution of productivity of each of the symmetric countries is a Frechet independent of n:

$$\lim_{t\to\infty} F(xe^{\nu t}, t) = e^{-\lambda x^{-\frac{1}{\theta}}}.$$

References

ALVAREZ, F., F. BUERA, AND R. LUCAS JR (2008): "Models of Idea Flows," *NBER Working Paper 14135.*

ARKOLAKIS, C., A. COSTINOT, D. DONALDSON, AND A. RODRÍGUEZ-CLARE (2012): "The Elusive Pro-Competitive Effects of Trade," *Manuscript, Yale University.*

BERNARD, A., J. EATON, J. JENSEN, AND S. KORTUM (2003): "Plants and Productivity in International Trade," *The American Economic Review*, 93, 1268–1290.

EMBRECHTS, P., C. KLUPPELBERG, AND T. MIKOSCH (2003): *Modelling Extremal Events: for Insurance and Finance*, Springer.

HOLMES, T., W. HSU, AND S. LEE (2012): "Allocative Efficiency, Markups, and the Welfare Gains from Trade," *Manuscript, University of Minnesota.*

KORTUM, S. (1997): "Research, patenting, and technological change," *Econometrica: Journal of the Econometric Society*, 65, 1389–1419.