Online Appendix for "Behavioral Macroeconomics via Sparse Dynamic Programming"

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12 Some Other Applications

To make sure that the model is widely applicable, I developed a behavioral version of a few other important machines of dynamic economics.

12.1 Dynamic Portfolio Choice

I now study a Merton problem with dynamic portfolio choice. The agent's utility is: $\mathbb{E}\left[\frac{1}{1-\gamma}\int_0^\infty e^{-\rho s} c_s^{1-\gamma} ds\right]$, and his wealth w_t evolves according to:

$$dw_t = (-c_t + rw_t)\,dt + w_t\theta_t\,(\pi_t dt + \sigma dZ_t)$$

where π_t is the equity premium and θ_t the allocation to equities.

I start by describing the rational problem, and then the behavioral solution. I call $\psi = \frac{1}{\gamma}$ the IES. Although for simplicity I use a CRRA utility function, I try to write the expressions in a way that involves both γ and ψ, a way that would generalize correctly to Epstein-Zin utility, where the two notions are disentangled.

12.1.1 Taylor expansions of the value function: rational case

We examine the problem in the rational case first, with a reminder of notions of portfolio choice. In a deterministic context with interest rate r_t, the SDF is simply $M_t = e^{-\int_0^t r_s ds}$. Next, suppose that there is a stochastic opportunity set: set of assets with risk premium π_t and covariance matrix Σ_t. In a static maximization, the optimal portfolio of the certainty equivalent is a return: $R_t(\theta_t) = r_t + \theta_t\pi_t - \frac{\gamma}{2}\theta_t\Sigma_t\theta_t$, so that the (static) optimal portfolio choice is $\theta_t = \arg\max_\theta R_t(\theta)$, i.e. $\theta_t = \frac{1}{\gamma}\Sigma_t^{-1}\pi_t$, and the certainty equivalent is finally: $R_t = \max_{\theta_t} R_t(\theta_t)$

$$R_t = r_t + \frac{1}{2\gamma}\Lambda_t \tag{81}$$

where

$$\Lambda_t = \pi_t'\Sigma_t^{-1}\pi_t \tag{82}$$

the "squared Sharpe ratio" of the investment opportunity set. Suppose that the process is driven by a Brownian motion B_t (which may be multidimensional) - if the price of risk is λ_t (so that

$\Lambda_t = \|\lambda_t\|^2$), the stochastic discount factor can be represented as:

$$M_t = \exp\left[-\int_0^t \left(\left(r_s + \frac{\Lambda_t}{2}\right) ds - \lambda_s dB_s\right)\right] \tag{83}$$

The value function is as follows.

Lemma 12.1 (Value function, traditional case) *Suppose that the interest rate r_t and and the price of risk λ_t are deterministic, and that the agent is the traditional rational agent. The value function derivative is*

$$V_w\left(w_t, x_t\right) = \left(\mu_t w_t\right)^{-\gamma}$$

and the optimal policy is to consume $c_t = \mu_t w_t$ (μ_t is the MPC to consume out of wealth), where:

$$\mu_t^{-1} = \mathbb{E}_t\left[\int_0^\infty e^{-\psi\rho s}\left(\frac{M_{t+s}}{M_t}\right)^{1-\psi} ds\right] = \mathbb{E}_t\left[\int_0^\infty e^{-\int_t^s (\psi\rho_u + (1-\psi)R_u)du} dt\right] \tag{84}$$

where

$$R_t = r_t + \frac{1}{2\gamma}\Lambda_t.$$

is the certainty equivalent of expected portfolio returns (comprising stocks and bonds), with $\Lambda_t = \|\lambda_t\|^2$ is the square Sharpe ratio of the investment opportunity set.

When the opportunity set is constant, we have $R_t = R_*$ and $\mu_t = \mu_*$ with

$$\mu_* = \psi\rho + (1-\psi) R_*. \tag{85}$$

When it is not constant, we have, up to second order terms:

$$\mu_t = \psi\rho + (1-\psi) R_t \tag{86}$$

where $R_t = \mu_* V_t^R$ is the average future portfolio returns, and V_t^R is the present value of future portfolio returns.

$$V_t^R := \mathbb{E}_t\left[\int_t^\infty e^{-\mu_*(s-t)} R_s ds\right] \tag{87}$$

Here R_t is the future average return of the portfolio (including stocks and bonds). Hence, the marginal propensity to consume is a weighted average (with weights ψ and $1-\psi$) of the pure rate of time preference ρ and the average future return of the portfolio.

Lemma 12.1 summarizes and somewhat generalizes well-known notions, particularly from the work of Campbell and Viceira (2002). It indicates that what matters is the risk-adjusted rate of return of the portfolio, R_t: it is the safe short-term rate r_t, plus the square Sharpe ratio Λ_t, divided by two times the risk aversion. The future average return R_t is key to capture the (leading order of) the value function. Related ideas are found in Basak and Chabakauri (2010) and Malamud and

Vilkov (2015).

To structure the problem, suppose that the vector of asset returns $d\tilde{r}_t$ (where $d\tilde{r}_{it}$ is the return of asset i) :

$$d\tilde{r} = (r + \pi_* + \hat{\pi}_t)\, dt + \sigma dZ_t$$
$$\hat{\pi}_t = f' X_t$$

where X_t is a vector of factors, following an AR(1):[74]

$$dX_t = -\Phi X_t dt + \sigma^X dZ_t$$

and f is a matrix of weights. We call

$$\Sigma^{r,X} = cov\left(d\tilde{r}, dX_t'\right)/dt = \sigma\sigma^{X\prime}.$$

the matrix of covariance, i.e. $\Sigma_{ij}^{r,X} = cov\left(d\tilde{r}_{it}, dX_{jt}\right)/dt$. We define $\theta_* = \frac{1}{\gamma}\Sigma_*^{-1}\pi_*$ as the portfolio choice in the model with constant variance and expected returns.

Then, the portfolio return is

$$\begin{aligned} R_t &= \frac{1}{2\gamma}\left(\pi_* + \hat{\pi}_t\right)' \Sigma_t^{-1}\left(\pi_* + \hat{\pi}_t\right) = \frac{1}{2\gamma}\pi_*\Sigma_*^{-1}\pi_* + \theta_*'\hat{\pi}_t + O\left(\|X_t\|^2\right) \\ &= R_* + \theta_*'\hat{\pi}_t \\ &= R_* + \theta_*' f' X_t = R_* + b' X_t \end{aligned}$$

i.e. the return is augmented by $\theta_*'\hat{\pi}_t$, with

$$b := f\theta_*.$$

Then, the present value of returns (87) is

$$V_t^R = \frac{R_*}{\mu_*} + b'\left(\mu_* I + \Phi\right)^{-1} X_t \tag{88}$$

where I is the identity matrix of the X's dimension.

For instance, if X_t is one-dimensional, then $bX_t = \hat{R}_t := R_t - R_*$, and $R_t := R_* + \frac{\mu_*}{\mu_*+\Phi}\hat{R}_t$.

$$\mu_t = \mu_* + (1-\psi)\frac{\mu_*}{\mu_* + \Phi}\hat{R}_t \tag{89}$$

Hence, we obtain a tractable representation of the value function to the leading order.

[74] Or X_t could be a linearity-generating twisted-AR(1), so that the derivations below can be exact (Gabaix 2009).

12.1.2 The hedging demand

We can calculate the hedging demand.

Lemma 12.2 (Hedging demand, rational) *The stock demand is*

$$\theta_t = \frac{1}{\gamma}\Sigma_t^{-1}\left(\pi_t + H_t\right) \tag{90}$$

where H_t is the hedging demand premium, equal to (up to second order terms):

$$H_{it} = (1-\gamma)\, cov\left(d\tilde{r}_i, dV_t^R\right) \tag{91}$$

i.e. H_{it} is $(1-\gamma)$ times the covariance between asset i's return $(d\tilde{r}_i)$ and the present value of future returns V_t^R (equation 87).

In the AR(1) framework above,

$$H_t = (1-\gamma)\,\Sigma^{r,X}\left(\mu_* I + \Phi'\right)^{-1} b. \tag{92}$$

Suppose that returns mean-revert, i.e. $cov\left(d\tilde{r}_{it}, d\frac{R_t}{\mu_*}\right) < 0$. So, if $\psi < 1$, then investors load more on stocks because of the hedging demand.

We next state the modification of the value function.

Lemma 12.3 (Value function with hedging demand, rational) *In the hedging demand context, we have:*

$$\mu_t = \psi\rho + (1-\psi)\left(R_t + \theta' H_t\right) \tag{93}$$

where $R_t = \mu_* V_t^R$ is the expected present value of returns, and H_t is the hedging demand term; they are explicit in (88) and (92).

The intuition for (90) is that H_{it} is a risk-adjusted risk premium of asset i. This intuition carries over to (93). Compared to (86), the expression for $\mu\left(X_t\right)$ offers one more term, the term $(1-\psi)\,\theta' H_t$.

A tractable case The equity premium $\pi_t = \overline{\pi} + \widehat{\pi}_t$ has a variable part $\widehat{\pi}_t$, which follows

$$d\widehat{\pi}_t = -\phi_R\widehat{\pi}_t dt - \chi_t\sigma dZ_t^1 + \sigma_\pi' dZ_t^2$$

where the return is $d\tilde{r}_t = \left(r_t + \pi_t\right)dt + \sigma dZ_t^1$. The parameter $\chi_t \geq 0$ indicates that equity returns mean-revert: good returns today lead to lower returns tomorrow. That will create a hedging demand term.

We call $\theta_* := \frac{\overline{\pi}}{\gamma\sigma^2}$ the standard, myopic demand for stocks.

12.1.3 The sparse agent's investment and consumption

We can calculate the sparse agent's demand. Recall that $\psi = 1/\gamma$ is the IES. We state again the proposition.

Proposition 12.4 *(Behavioral dynamic portfolio choice) The fraction of wealth allocated to equities is, with $\theta_* := \frac{\overline{\pi}}{\gamma\sigma^2}$,*

$$\theta_t^s = \theta_* + \tau\left(\frac{\widehat{\pi}_t}{\gamma\sigma^2}, \kappa\right) + \tau\left(\frac{H_t}{\gamma\sigma^2}, \kappa_\theta\right)$$

while consumption is $c_t^s = \mu_t^s w_t$ with

$$\mu_t^s = \mu_* + \tau\left((1-\psi)\frac{\mu_*}{\mu_* + \Phi}\theta_*\widehat{\pi}_t, \kappa_{c/w}\right) + \tau\left((1-\psi)\,\theta_* H_t, \kappa_{c/w}\right)$$

where H_t is the hedging demand term (94)

$$H_t = (1-\gamma)\, cov\left(dr_t, dV_t^R\right) = -(1-\gamma)\,\theta_*\frac{1}{\mu_* + \Phi}\sigma^2\chi_t$$

Proof We first calculate the rational values. In that case

$$R_t = r_* + \frac{\Lambda_*}{2\gamma} + \theta_*\frac{\mu_*}{\mu_* + \Phi}\widehat{\pi}_t$$

$$H_t = (1-\gamma)\, cov\left(dr_t, d\left(\frac{R_t}{\mu_*}\right)\right) = -(1-\gamma)\,\theta_*\frac{1}{\mu_* + \Phi}\sigma^2\chi_t \tag{94}$$

so that

$$\theta_t = \frac{\pi_* + \hat{\pi}_t + H_t}{\gamma\sigma^2}$$

In addition

$$\mu_t = \psi\rho + (1-\psi)\left(R_t + \theta_*' H_t\right) = \mu_* + (1-\psi)\left(\theta_*\frac{\mu_*}{\mu_* + \Phi}\widehat{\pi}_t + \theta_*' H_t\right)$$

As in Proposition 3.9, with ex-post attention, the BR agent just truncates those terms.
□

Proposition 12.4 predicts the choice of a sparse agent. When $\kappa = 0$, it is the policy of a fully rational agent, e.g. as in Campbell and Viceira (2002). When $\kappa > 0$, it is the policy of a sparse agent. When κ is larger, portfolio choice becomes insensitive to the change in the equity premium, $\widehat{\pi}_t$, and the agent thinks less about the mean-reversion of asset, the $B\chi$ terms.

In addition, the agents' consumption function pays little attention to the mean-reversion of assets.

12.1.4 Proofs for the Merton portfolio problem

Proof of Lemma 12.1 Here we present a proof sketch, in part because those notions are well-known. We record the values with a time-discounting of D_t, with $D_t = e^{-\rho t}$ in the infinite horizon, but D_t could be different to capture finite-time horizon effects. For instance, with a finite horizon of T, and a terminal weight b on the last consumption, then $D_t = ae^{-\rho t}1_{t\leq T} + b\delta(t-T)$.

First, in the SDF approach, the problem is

$$\max \mathbb{E}\int_0^\infty D_t c_t^{1-\gamma} dt \text{ s.t. } \mathbb{E}\int_0^\infty M_t c_t dt = w_0$$

This leads to $Dc_t^{-\gamma} = k' M_t$ and $c_t = kD_t^{\psi} M_t^{-\psi}$ for constant k, k'. The constant is determined by the budget constraint , $w_0 = \mathbb{E}\int_0^\infty M_t c_t dt = k\mathbb{E}\int_0^\infty kD_t^{\psi} M_t^{1-\psi}$. This leads to a utility derivative $V_w = (\mu_0 w_0)^{-\gamma}$, with

$$\mu_0^{-1} = \mathbb{E}\left[\int_0^\infty D_t^{\psi} M_t^{1-\psi} dt\right] \tag{95}$$

When M_t follows (83), routine calculations show that

$$\mu_0^{-1} = \mathbb{E}\left[\int_0^\infty D_t^{\psi} e^{-(1-\psi)\int_0^t R_u du} dt\right]$$

We next proceed to a Taylor expansion:

$$\begin{aligned}
\mu_0^{-1} &= \mathbb{E}\left[\int_0^\infty D_t^{\psi} e^{-(1-\psi)\int_0^t \left(R_* + \hat{R}_u\right) du} dt\right] \\
&= \mathbb{E}\left[\int_0^\infty D_t^{\psi} e^{-(1-\psi) R_* t}\left(1 - (1-\psi)\int_0^t \hat{R}_u du\right) dt\right]
\end{aligned}$$

With an infinite horizon, $D_t = e^{-\rho t}$ and

$$\begin{aligned}
\mu_0^{-1} &= \mathbb{E}\left[\int_0^\infty e^{-\mu_* t}\left(1 - (1-\psi)\int_0^t \hat{R}_u du\right) dt\right] = \frac{1}{\mu_*} - (1-\psi)\,\mathbb{E}\left[\int_{t=0}^\infty e^{-\mu_* t}\int_0^t \hat{R}_u du dt\right] \\
&= \frac{1}{\mu_*} - (1-\psi)\,\mathbb{E}\left[\int_0^t \left(\int_{t=u}^\infty e^{-\mu_* t}\right)\hat{R}_u du\right] = \frac{1}{\mu_*} - (1-\psi)\,\mathbb{E}\left[\int_0^t \frac{1}{\mu_*} e^{-\mu_* u}\hat{R}_u du\right] \\
&= \frac{1}{\mu_*} - (1-\psi)\frac{1}{\mu_*^2}\mathbb{E}\left[\int_0^t \mu_* e^{-\mu_* u}\hat{R}_u du\right] \\
&= \frac{1}{\mu_*} - (1-\psi)\frac{1}{\mu_*^2}(R_0 - R_*) \text{ with } R_0 - R_* = \mathbb{E}\left[\int_0^t \mu_* e^{-\mu_* u}\hat{R}_u du\right] \\
&= \frac{1}{\mu_* + (1-\psi)(R_0 - R_*)} + O(R_0 - R_*)^2.
\end{aligned}$$

so

$$\mu_0 = \mu_* + (1-\psi)(R_0 - R_*) = \psi\rho + (1-\psi)R_* + (1-\psi)(R_0 - R_*)$$
$$= \psi\rho + (1-\psi)R_0$$

When the consumer has a finite horizon and only cares about date T consumption, then $D_t = \delta(t-T)$, and

$$\mu_0^{-1} = \mathbb{E}\left[\int_0^\infty D_t^\psi e^{-(1-\psi)R_* t}\left(1-(1-\psi)\int_0^t \hat{R}_u du\right)dt\right]$$
$$= e^{-\mu_* T} - e^{-\mu_* T}(1-\psi)\,\mathbb{E}\left[\int_0^T \hat{R}_u du\right]$$

so the MPC is 0 but we have

$$\mu_t^{-1} = e^{-\mu_*(T-t)}\left(1-(1-\psi)\,\mathbb{E}\left[\int_t^T \hat{R}_u du\right]\right) \tag{96}$$

so again μ_t is related to the present value of future portfolio returns.

□

Proof of Lemma 12.2 In semi-discrete notation the asset demand at time t comes from

$$\max_\theta \mathbb{E}_t\left[V\left(w\left(1+r_t dt + \theta d\tilde{r}_t\right), X_t + dX_t\right)\right]$$

where, with $\pi_t = \pi_* + f'X_t$,

$$\mathbb{E}_t\left[dV_t\right] = \mathbb{E}_t\left[V\left(w\left(1+r_r dt + \theta d\tilde{r}_t\right), X_t + dX_t\right) - V\left(w, X_t\right)\right]$$
$$= V_w w\left(r_t + \theta'\pi_t\right)dt + V_{wX} w\left\langle \theta' d\tilde{r}_t, dX_t\right\rangle + V_{ww} w^2 \theta'\Sigma_t\theta dt + \frac{1}{2}Tr\left(V_{XX}\Sigma^{X,X}\right)dt$$
$$= V_w w\left[\theta'\left(\pi_t + H_t\right) - \frac{\gamma}{2}\theta'\Sigma_t\theta dt\right]dt + \frac{1}{2}Tr\left(V_{XX}\Sigma^{X,X}\right)dt$$

where

$$\theta' H_t = \frac{V_{wX}}{V_w}\left\langle \theta d\tilde{r}_t, dX_t\right\rangle$$

is the hedging demand premium term. This implies

$$\theta = \frac{1}{\gamma}\Sigma_t^{-1}\left(\pi_t + H_t\right)$$

To calculate H_t more fully, recall that $V_w = \mu(X_t)^{-\gamma} w^{-\gamma}$, so that $\ln V_w = -\gamma\mu(X_t) - \gamma \ln w$, and

$$\frac{V_{wX}}{V_w} = -\gamma \frac{\mu_X}{\mu_*} = -\gamma(1-\psi)\frac{R_X}{\mu_*} = (1-\gamma)\frac{R_X}{\mu_*} =: B'$$

with

$$\frac{R_X}{\mu^*} = b'(\mu_* I + \Phi_R)^{-1}$$

Note that $R_t = r + \frac{1}{2\gamma}\pi_t' \Sigma^{-1}\pi_t$ with $\pi_t = \pi_* + \hat{\pi}_t$, so, with $R_* = r + \frac{1}{2\gamma}\pi_*'\Sigma^{-1}\pi_*$

$$R_t = R_* + \frac{1}{\gamma}\pi_*'\Sigma^{-1}\hat{\pi}_t = R_* + \theta_*'\hat{\pi}_t$$
$$R_t = R_* + \theta_*' f' X_t \tag{97}$$

hence

$$b = f\theta \tag{98}$$

Hence,

$$\theta' H_t = \frac{V_{wX}}{V_w}\langle \theta d\tilde{r}_t, dX_t\rangle = \sum_{i,j}\theta_i \langle d\tilde{r}_{it}, B_j dX_j\rangle = \theta_i \Sigma_{ij}^{r,X} B_j$$

so that

$$H_t := \Sigma^{r,X} B = (1-\gamma)\Sigma^{r,X}\frac{\bar{R}_X'}{\mu_*} = (1-\gamma)\, cov\left(d\tilde{r}, d\frac{R_t}{\mu_*}\right)$$
$$= (1-\gamma)\Sigma^{r,X}(\mu_* I + \Phi_R')^{-1} b$$

Proof of Lemma 12.3 Suppose

$$d\tilde{r} = (r + \pi_* + fX_t)\,dt + \sigma dZ_t$$

and that agents have a constant MPC μ_* :

$$\frac{dw_t}{w_t} = (r - \mu_*)\,dt + \theta' d\tilde{r} = (r - \mu_* + \theta\pi_* + \theta' f' X_t)\,dt + \theta\sigma dZ_t$$
$$= (g_* + \theta' f' X_t)\,dt + \theta\sigma dZ_t$$
$$\frac{dw_t}{w_t} = (g_* + b' X_t)\,dt + \theta\sigma dZ_t$$

with $b = f\theta$ and

$$g_* := r + \theta\pi_* - \mu_*$$

We want to calculate (assuming the policy $c_t = \mu_* w_t$, which leads only to second order losses)

$$U = \mathbb{E}\left[\frac{1}{1-\gamma}\int_0^\infty e^{-\rho s} c_s^{1-\gamma} ds\right] = \frac{\mu_*^{1-\gamma}}{1-\gamma}\mathbb{E}\left[\int_0^\infty e^{-\rho s} w_s^{1-\gamma} ds\right]$$

Calling

$$m_s = e^{-\rho s} w_s^{1-\gamma}$$

We calculate

$$\begin{aligned}\frac{dm_t}{m_t} &= -\rho + (1-\gamma)\left(g_* + b' X_t - \frac{\gamma}{2}\|\theta\sigma\|^2\right) dt + (1-\gamma)\,\theta'\sigma dZ_t \\ &= (-a + (1-\gamma)\, b' X_t)\, dt + (1-\gamma)\,\theta'\sigma dZ_t\end{aligned}$$

$$\begin{aligned}a &:= \rho - (1-\gamma)\left(g_* - \frac{\gamma}{2}\|\theta\sigma\|^2\right) \\ &= \rho - (1-\gamma)\left(r + \theta\pi_* - \mu_* - \frac{\gamma}{2}\|\theta\sigma\|^2\right) = \rho - (1-\gamma)\,(R_* - \mu_*) \\ &= \mu_*\end{aligned}$$

We calculate linearly generating (LG moments. We assume $dX_t = -\Phi X_t dt + \sigma^X dZ_t + O\left(\|X_t\|^2\right)$:

$$\begin{aligned}\mathbb{E}\left[\frac{dm_t}{m_t}\right]/dt &= -\mu_* + (1-\gamma)\, b' X_t \\ \mathbb{E}\left[\frac{d\,(m_t X_t)}{m_t}\right]/dt &= (-\mu_* + (1-\gamma)\, b' X_t)\, X_t - \Phi X_t + (1-\gamma)\,\langle \theta' d\tilde{r}_t, dX_t\rangle \\ &= (1-\gamma)\,\theta'\,\langle d\tilde{r}_t, dX_t\rangle + (-\mu_* - \Phi)\, X_t + O\left(\|X_t\|^2\right)\end{aligned}$$

so the LG generator (Gabaix, 2009) is

$$\omega = \begin{pmatrix} \mu_* & -(1-\gamma)\, b' \\ -(1-\gamma)\,\Sigma^{X,r}\theta & \mu_* + \Phi \end{pmatrix}$$

Hence, the present value is $V = (1,0)\,\omega^{-1}\cdot(1, X_t)$

We use the formula for the inversion of the block matrix:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1} BD^{-1} \\ * & * \end{pmatrix}$$

where $*$ are terms we will not use. We have

$$(1,0)\,\omega^{-1} = \left(f, f\,(1-\gamma)\,b'\,(\mu_* + \Phi)^{-1}\right)$$
$$f := \left(\mu_* - (1-\gamma)^2\,b'\,(\mu_* + \Phi)^{-1}\,\Sigma^{X,r}\theta\right)^{-1}$$

so

$$V = f\left(1 + (1-\gamma)\,b'\,(\mu_* + \Phi)^{-1}\,X_t\right) \tag{99}$$

The value function has the Taylor expansion: $V(w_t, X_t) = v(X_t)\,\mu_*^{1-\gamma}\frac{w_t^{1-\gamma}}{1-\gamma}$

$$\begin{aligned}
v(X_t) &= \frac{1 + (1-\gamma)\,b'\,(\mu_* + \Phi)^{-1}\,X_t}{\mu_{**}} \\
\mu_{**} &= \mu_* - (1-\gamma)^2\,b'\,(\mu_* I + \Phi)^{-1}\,\Sigma^{X,r}\theta \\
&= \mu_* - \frac{(1-\gamma)^2}{\left(1-\frac{1}{\psi}\right)}H_t'\theta \text{ using (92), } H_t = \left(1 - \frac{1}{\psi}\right)\Sigma^{r,X}\,(\mu_* I + \Phi')^{-1}\,b \\
&= \mu_* - (1-\gamma)\,H_t'\theta
\end{aligned}$$

Rewrite

$$\begin{aligned}
V &= v(X_t)\,\mu_*^{1-\gamma} = \frac{1+K}{\mu_* + L}\mu_*^{1-\gamma} \text{ with} \\
K &= (1-\gamma)\,b'\,(\mu_* + \Phi)^{-1}\,X_t \\
L &= -(1-\gamma)\,H_t'\theta \\
V &= (\mu_* + C)^{-\gamma} = \mu_*^{-\gamma}\left(1 - \gamma\mu_*^{-1}C\right) \\
&= \mu_*^{-\gamma}\left(1 + K - \mu_*^{-1}L\right)
\end{aligned}$$

Hence,

$$\begin{aligned}
\mu_t - \mu_* = C &= -\frac{\mu_*}{\gamma}K + \frac{1}{\gamma}L = (1-\psi)\,b'\mu_*\,(\mu_* + \Phi)^{-1}\,X_t + \frac{-1}{\gamma}\,(1-\gamma)\,H_t'\theta \\
&= (1-\psi)\,b'\mu_*\,(\mu_* + \Phi)^{-1}\,X_t + (1-\psi)\,H_t'\theta
\end{aligned}$$

$$\mu^I := \frac{1}{\gamma}L = \frac{-1}{\gamma}\,(1-\gamma)\,H_t'\theta = \left(1 - \frac{1}{\psi}\right)H_t'\theta$$

Intuition: the extra present value of returns is

$$\begin{aligned}
\frac{R_t - R_*}{\mu_*} &= \frac{C}{(1-\psi)\,\mu_*} = b'\,(\mu_* + \Phi)^{-1}\,X_t - \psi\,(1-\psi)\,\frac{1}{\mu^*}b'\,(\mu_* I + \Phi)^{-1}\,\theta\Sigma^{r,X} \\
&= b'\,(\mu_* + \Phi)^{-1}\left(X_t + (1-\gamma)\,\frac{1}{\mu^*}\theta\Sigma^{r,X}\right)
\end{aligned}$$

12.2 Linear-Quadratic models

Many economic problems can be conveniently expressed as linear-quadratic (LQ) models (Ljungqvist and Sargent 2012). We show here how to systematically derive a BR version of those models.

We again write $z = (w, x)$, where w is the set of variables known under the default model, and x is the set of variables that are not considered in the default model. Utility is:

$$u(z, a) := \frac{1}{2} \begin{pmatrix} z \\ a \end{pmatrix}' \begin{pmatrix} U_{zz} & U_{za} \\ U_{az} & U_{aa} \end{pmatrix} \begin{pmatrix} z \\ a \end{pmatrix}$$

and the law of motion is:

$$z' = F^z(z, a) := \Gamma_z^z z + \Gamma_a^z a$$

where U and Γ are constant matrices. The rational value function is also LQ

$$V(z) = -\frac{1}{2} z' V_{zz} z = \frac{-1}{2} \left(w' V_{ww} w + 2 w' V_{wx} x + x' V_{xx} x \right)$$

Under the default model V_{ww} is known, and

$$a^d(w) = A_w w$$

for A_w a constant. Our goal is to find V_{wx}, which affects the value function. To do so, we apply from (329).

Lemma 12.5 *In the linear-quadratic problem, the cross-partial derivative of the value function is*

$$V_{wx} = V_{wx} = \left[1 - \beta \left(D_w w'\right) \cdot \Gamma_x^{x'}\right]^{-1} \left[U_{xw} + U_{xa} A_w + \beta \Gamma_x^w V_{ww} \left(D_w w'\right)\right].$$

where $D_w w' = \Gamma_w^w + \Gamma_a^w A_w$. *The impact on the action is* $a = A_w w + A_x x$, *where* A_w *is the default value, and*

$$A_x = -\Psi_a^{-1} \Psi_x, \tag{100}$$

where

$$\Psi_a = U_{aa} + \beta \Gamma_a^w V_{ww} \Gamma_a^w$$
$$\Psi_x = U_{xa} + \beta V_{wx} \Gamma_a^w$$

This illustrates that the value function can be written:

$$V(z) = -\frac{1}{2} z' V_{zz} z = \frac{-1}{2} w' V_{ww} w + w' V_{wx} x + O\left(\|x\|^2\right)$$

with matrix V_{wx} as expressed in closed form above.

Hence, the BR value function is simply:

$$V^s(z,m) = -\frac{1}{2}z'V_{zz}z = \frac{-1}{2}w'V_{ww}w + w'V_{wx}M(m)x + O\left(\|x\|^2\right)$$

for the diagonal attention matrix $M(m) = diag(m_{x_i})$.

Proposition 12.6 (Behavioral version of linear-quadratic problems) *In a linear-quadratic problem, the optimal attention is*

$$m_{x_i} = \mathcal{A}\left(A_{x_i}\Psi_a A_{x_i}\sigma_i^2/\kappa\right) \tag{101}$$

and the optimal sparse action is

$$a = A_w w + A_x M x$$

where $M = diag(m_{x_i})$. Here we use the notations of Lemma 12.5.

12.3 Precautionary saving

The consumer may save more when the future is uncertain, a phenomenon known as "precautionary savings." This is easy to obtain in this BR model. Suppose that the income process is $\hat{y}_{t+1} = \rho_y\hat{y}_t + \varepsilon_{t+1}$, for ε a mean-0 variable. Then, the rational value function does not obtain in closed form, unless we assume very specific functional forms (CARA utility, Gaussian noise). What to do then?

Let us first derive the rational policy.[75]

Lemma 12.7 (Rational policy with precautionary saving) *With stochastic income shocks, the rational value function is*

$$\begin{aligned}
V\left(w_t,\hat{y}_t,\sigma_\varepsilon^2\right) &= \frac{R}{r}\mathbb{E}u\left(\frac{\bar{r}}{R}w + \bar{y} + \frac{\bar{r}}{R}\sum_{\tau>t}\frac{\hat{y}_\tau}{R^{\tau-t}}\right) + o\left(\sigma_\varepsilon^2\right) \\
&= \frac{R}{r}u\left(\frac{\bar{r}}{R}w + \bar{y} + \frac{\bar{r}}{R}\mathbb{E}\left[\sum_{\tau>t}\frac{\hat{y}_\tau}{R^{\tau-t}}\right] - \frac{\Gamma}{2}\left(\frac{\bar{r}}{R}\right)^2 var_t\left(\sum_{\tau>t}\frac{\hat{y}_\tau}{R^{\tau-t}}\right)\right) + o\left(\sigma_\varepsilon^2\right)
\end{aligned}$$

where $\Gamma = -\frac{u''(c^d)}{u'(c^d)}$ $\left(c^d = \frac{\bar{r}}{R}w_t + \bar{y}\right)$ is the coefficient of absolute risk aversion. For instance, with an AR(1) process:

$$\hat{c}_t = \frac{\bar{r}}{R}\frac{\hat{y}_t}{R-\rho_y} - \frac{\Gamma}{2}\left(\frac{\bar{r}}{R}\right)^2\frac{var\left(\varepsilon_{t+1}^y\right)}{\left(R-\rho_y\right)^2} \tag{102}$$

Then, the agent may, or may not, take the noise into account.

[75] See Wang, Wang and Yang (2013) and Achdou et al. (2015) for recent analytical progress on this issue.

Proof First, observe that he two RHS are identical, up to $o\left(\sigma_{\varepsilon}^{2}\right)$ terms.

To see the second statement, let us take more successively more complex problems. First, for small noise, we have, as in Arrow-Prat:

$$\mathbb{E}\left[u\left(X\right)\right]=u\left(\mathbb{E}\left[X\right]-\frac{\Gamma}{2}var\left(X\right)\right)+o\left(var\left(X\right)\right)$$

with $\Gamma=-\frac{u''(0)}{u'(0)}$.

Second, suppose that $\hat{y}_{t+1}=\rho\hat{y}_{t}+\varepsilon_{t+1}^{y}$. We are looking for an expansion of the type $V\left(w,\hat{y},\sigma_{y}^{2}\right)=V\left(w+A,\hat{y},0\right)=V^{d}\left(w+A+\frac{\hat{y}}{R-\rho_{y}}\right)$, for some A. We have

$$V^{d}\left(w+A,\hat{y}\right)=\max_{c}u\left(c\right)+\beta\mathbb{E}\left[V^{d}\left(R\left(w-c\right)+y+A,\mathbb{E}\left[\hat{y}_{t+1}\right]+\varepsilon_{t+1}^{y}\right)\right]$$

so, taking the Taylor expansion of $\widehat{y}_{t+1}$ around $\mathbb{E}_{t}\widehat{y}_{t+1}$

$$V_{w}^{d}A=\beta V_{w}^{d}A+\frac{1}{2}\beta V_{yy}^{d}var\left(\varepsilon_{t+1}^{y}\right)$$

so, using $\beta=\frac{1}{R}$

$$\begin{aligned}A&=\frac{1}{2}\frac{\beta}{1-\beta}\frac{V_{yy}^{d}}{V_{w}^{d}}var\left(\varepsilon_{t+1}^{y}\right)\\&=-\frac{1}{2}\frac{\bar{r}}{R}\Gamma^{V^{d,0}}\frac{1}{\left(R-\rho_{y}\right)^{2}}var\left(\varepsilon_{t+1}^{y}\right)\end{aligned}$$

and as $\Gamma^{V^{d,0}}=\frac{\bar{r}}{R}\Gamma^{u}$,

$$A=-\left(\frac{\bar{r}}{R}\right)^{2}\frac{\Gamma^{u}}{2}\frac{1}{\left(R-\rho_{y}\right)^{2}}var\left(\varepsilon_{t+1}^{y}\right).$$

Next, for a more general process with state vector z, we have, by the same reasoning,

$$A=\frac{1}{2}\frac{\bar{r}}{R}\frac{\mathbb{E}\left[\varepsilon_{z}V_{zz}^{d}\varepsilon_{z}\right]}{V_{w}^{d}}$$

Now, it is not trivial to get V_{zz}. Indeed, we have $V\left(w,z\right)=V^{d}\left(w+b\cdot z\right)+O\left(z^{2}\right)$, but that expression gives only part of V_{zz}. The "certainty equivalent" approach works well for income shocks, but not for uncertainty about interest rates, say.□

A sparse agent will, in contrast, do

$$\hat{c}_{t}=m_{y}\frac{\bar{r}}{R}\frac{\hat{y}_{t}}{R-\rho_{y}}-m_{\sigma_{y}^{2}}\frac{\Gamma}{2}\left(\frac{\bar{r}}{R}\right)^{2}\frac{var\left(\varepsilon_{t+1}^{y}\right)}{\left(R-\rho_{y}\right)^{2}}$$

with some inattention to risk $m_{\sigma_{y}^{2}}$. Hence, he will create a too small buffer of savings, compared to a rational agent.

Proposition 12.8 *A sparse agent saves too little against idiosyncratic shocks, compared to a rational agent.*

12.4 Investment Problems

Suppose that the agent needs to solve:

$$V(K_0) = \max_{I_t} \int e^{-\rho t} (\pi(K_t) - G(I_t))\, dt \text{ s.t. } \dot{K}_t = -\delta K_t + I_t$$

where $\pi(K)$ is the profit rate, $G(I) = I + g(I)$ is the cost of investment, inclusive of adjustment cost I. Capital depreciates at a rate δ.

How will the agent proceed? We apply the generalized k/K procedure of section 15.8, using $(X, a) := (K, I)$. As $u(K, I) = \pi(K) - g(I)$ we have

$$\bar{u}(k, K, I) = \pi(K) + \pi_K(K)(k - K) - g(I)$$

and $\bar{F}(k, K, I) = -\delta k + I$, so that

$$\dot{k}_t = -\delta k_t + I_t$$

Hence, the agent simply solves a model with linear profitability of capital:

$$V(k_0, K_0) = \max_{I_t} \int e^{-\rho t} (\pi_K(K_t) k_t - G(I_t))\, dt \text{ s.t. } \dot{k}_t = -\delta k_t + I_t$$

Hence, optimal investment satisfies

$$G'(I_0) = V_k(k_0, K_0) = \mathbb{E}^s \int_0^\infty e^{-(\rho+\delta)t} \pi_K(K_t)\, dt$$

i.e. on the RHS with have the subjective expectation of marginal profits.

At the steady state, with $K_t = K_*$, the optimum is characterized by $G'(I_*) = \pi_K(K_*)/(\rho + \delta)$ with $I_* = \delta K_*$ – as in the traditional model. As in the general procedure, I assume that the agent perceives a linear mean-reversion of the state variable: $\dot{K}_t = -\Phi K_t$, for some perceived speed Φ.

Outside the steady state,

$$\hat{I}_0 = \frac{\int_0^\infty e^{-(\rho+\delta)t} \pi_{KK}(K_*) \widehat{K}_t dt}{G''(I_*)} = -\int_0^\infty e^{-(\rho+\delta)t} \xi e^{-\Phi t} \widehat{K}_0 dt = -\frac{\xi}{\rho + \delta + \Phi} \hat{K}_0 = \frac{1}{G''(I_*)} \frac{\widehat{\pi_{K|t=0}}}{\rho + \delta + \Phi}$$

with $\xi := \frac{-\pi_{KK}(K_*)}{G''(I_*)} \geq 0$ and $\widehat{\pi_K} = \pi_{KK}\hat{K}_t$. Here,

$$\hat{I}_0 = \frac{1}{G''(I_*)} \frac{\widehat{\pi_{K|t=0}}}{\rho + \delta + \Phi}$$

means that the agent's investment reacts to current (marginal) profitability $\widehat{\pi_{K|t=0}}$, with a dampening indexed by Φ, which needs not be the "rational" amount of dampening. For instance, if Φ is low, the agent will overreact to current profitability. This shows, I hope, that the procedure is reasonable psychologically.

The investment policy $\hat{I}_t = \xi \frac{1}{r+\delta+\Phi} \hat{K}_t$ implies that the true law of motion of capital is

$$\hat{K}_t = -\delta \hat{K}_t + \hat{I}_t = -\left(\delta + \frac{\xi}{r + \delta + \Phi}\right) K_t$$

so that the true speed of mean-reversion is

$$\Phi^r = \delta + \frac{\xi}{r + \delta + \Phi}.$$

Under rational expectations, $\Phi = \Phi^r$, so that a fixed point needs to be solved for. In the more general model here, the agent perceives a speed of mean-reversion of profitability Φ, and reacts accordingly. De facto, he sets $G'' \cdot \hat{I}_0 = \mathbb{E}^s \int e^{-\rho t} \widehat{\pi_K} dt$, with $\widehat{\pi_K} = \pi_{KK} \hat{K}_t$, i.e. sets investment according to the perceived changes in future profitability of capital.

12.5 The Becker-Murphy model of Rational Addiction

The Becker-Murphy (1988) model of rational addiction is a peak of the use of rationality in economics. We will give a behavioral version of it. We shall see that the qualitative evidence in favor of the model (the fact that future increase in prices lower consumption today) are also consistent with this BR version - it shows that agent are at least *partially* rational (as in the present model), not that they are *fully* rational (as assumed by Becker-Murphy). This distinction is important: if people are BR enough, they'd be better off under a high tax, or a ban, of the addictive substance – while the optimal tax is 0 in the Becker-Murphy model. This analysis is in the spirit of Gruber and Kőszegi (2001), who study a hyperbolic discounting addict, rather than a boundedly rational one in the sense of this paper.

We call c the consumption and x the level of addition. Utility function is

$$u(c, x) = -\frac{1}{2}(c - x - A)^2 - Bx$$

Addition x_t evolves as

$$x_{t+1} = \rho x_t + h c_t.$$

The BR agent has in mind the model

$$x_{t+1} = \rho^s x_t + h^s c_t.$$

We posit that in the default model the agent does not perceive any addiction dynamics: he perceives addition as being constant.

$$\rho^d = 1, \qquad h^d = 0.$$

When the agent has partial attention m to inattention dynamics, we have

$$x_{t+1} = (1-m)\,x_t + m\,(\rho x_t + h c_t)$$

so

$$\rho^s = (1-m) + m\rho, \qquad h^s = mh$$

Let us now study the BR dynamics.

Warm-up: 2 period model As before, it is helpful to study a 2-period model, with $t = 1, 2$. *Behavior at the last period,* $t = 2$. The agent should and does consume his optimal consumption

$$c^d(x) = \arg\max_c u(c, x) = x + A$$

We define the resulting utility as $\bar{u}(x)$

$$\bar{u}(x) := \max_c u(c, x) = u\left(c^d(x), x\right) = -Bx.$$

To, the time-1 value function is

$$V^1(x) = \bar{u}(x). \tag{103}$$

Behavior at period 1, $t = 1$. Given perceived dynamics, the problem is

$$\operatorname*{smax}_{c;m} v(c, x, m)$$
$$v(c, x, m) := u(c, x) + \beta V(\rho^s(m)\,x + h^s(m)\,c)$$

which gives:

$$\begin{aligned} 0 &= u_c + \beta h^s V'(\rho^s x + h^s c) \\ &= -c + x + A - \beta h^s B \end{aligned}$$

$$c = x + A - \beta h^s B \tag{104}$$

An interesting variant is to impose $c \geq 0$. Then, first period consumption is > 0 iff $A - h^s B > 0$. So, if $h^s B < A \leq hB$, then *the rational agent consumes 0, while the very behavioral agent consumes a positive amount and becomes addicted.*

The optimal attention is $m = \mathcal{A}\left(v_{cc} c_m^2 / \kappa\right) = \mathcal{A}\left(-u_{cc} B^2 h^2 / \kappa\right)$.

Infinite horizon model The value function satisfies

$$V(x) = \operatorname*{smax}_{c;m} u(c,x) + \beta V\left(\rho^s(m) x + h^s(m) c\right)$$

The FOC is

$$u_c(c,x) + \beta V'\left(\rho^s(m) x + h^s(m) c\right) h^s(m) = 0$$

i.e. the agent takes into account only *part* of the addiction costs, as $h^s(m) \leq h$. As a result, the agent is more addicted in the steady state. The greater the myopia, the greater the optimal tax.

Proposition 12.9 *In the Becker-Murphy model with boundedly rational agents, the consumption c given the stock of addiction x is*

$$c(x) = x + A + \beta b(m) m^h h$$

using $m = \left(m^h, m^V\right)$; the value function is

$$V(x,m) = a(m) + b(m) x$$

where $b(m) = -\frac{B}{1-\beta\left(1+m^V(\rho+h-1)\right)}$ and $a(m)$ is in the proof. When using the plain (as opposed to iterated) sparse max, $m^V = 0$ and attention to addition is $m^h = \mathcal{A}\left(\frac{1}{\kappa}\left(\frac{\beta B h}{1-\beta}\right)^2\right)$.

Proof of Proposition 12.9 We're looking for a solution of the form: $V(x) = a + bx$, for a, b to be determined. The FOC is: $u_c + \beta V_x h^s = 0$, i.e. $-(c - x - A) + \beta b h^s = 0$ and

$$c = x + A + \beta b^s h^s$$
$$u(c(x), x) = -\frac{1}{2}\left(\beta b^s h^s\right)^2 - Bx$$

The self-consistency condition is

$$V(x) = u(c(x), x) + \beta V(\rho x + hc)$$

i.e.

$$a + bx = \frac{-1}{2}\left(\beta b^s h^s\right)^2 - Bx + \beta\left[a + b\left(\rho x + h\left(x + A + \beta b^s h^s\right)\right)\right]$$

This gives

$$b = \frac{-B}{1-\beta\overline{\rho}}$$
$$a = \beta\frac{hA + \beta b^s h h^s - \frac{1}{2}\beta\left(b^s h^s\right)^2}{1-\beta}$$

When the agent perceives $\rho' = 1 - m^V + m^V\rho$ and $h' = m^V h$ when forming the value function, we have the same expressions,

$$b^s(m) = \frac{-B}{1-\beta(\rho'+h')} = \frac{-B}{1-\beta\left(1+m^V(\rho+h-1)\right)}$$
$$a(m) = \beta\frac{hA + \beta b^s(m) h h^s - \frac{1}{2}\beta\left(b^s(m) h^s\right)^2}{1-\beta}$$

To determine optimal attention m, observe that in the 1-step smax, at the beginning, $m^V = 0$, so the perceived value function is

$$V\left(x, m^V = 0\right) = u(c(x), x) + \beta V\left(x, m^V = 0\right)$$

so

$$V\left(x, m^V = 0\right) = \frac{u(c(x), x)}{1-\beta} = \frac{-\frac{1}{2}\left(\beta b^s h^s\right)^2 - Bx}{1-\beta}$$

This implies: $b^s = -\frac{B}{1-\beta}$, and

$$c = x + A + \beta b^s\left(m^V = 0\right) hm$$
$$= x + A - \frac{\beta B}{1-\beta} hm$$

so that the impact of thinking more about h, while keeping the future value function constant is

$$\frac{\partial c}{\partial m} = \beta b^s\left(m^V = 0\right) h = -\frac{\beta B h}{1-\beta}$$

Hence, optimal attention is:

$$m = \mathcal{A}\left(\frac{1}{\kappa}\left(\frac{\partial c}{\partial m}\right)^2 u_{cc}\right) = \mathcal{A}\left(\frac{1}{\kappa}\left(\frac{\beta B h}{1-\beta}\right)^2\right)$$

12.6 Ricardian Equivalence: Reaction to taxes over time

For simplicity, I use continuous time. The interest rate is $r = -\ln\beta$. The government needs to collect a present value of G/r. This could be done by taxing the population (of size normalized to

1) by $H = Ge^{rT}$, starting at a period T.[76] Hence, the path of taxes is 0 for $t < T$, and H for $t \geq T$.

What is a consumer's response at time $t < T$? If the consumer is perfectly attentive, then he should start saving at time 0. However, a sparse agent might not pay attention to those future taxes increases and start cutting on consumption only later, perhaps, just when the tax cuts are enacted.

Let us analyze this more in detail. At T, the tax H is enacted, so that for $t \geq T$, the agent is aware of it. This yields consumption deviation from the default value: $\widehat{c}_t = r\widehat{w}_t - H$.

Before the enactment of taxes ($t < T$), will the consumer think of the tax H? That tax lowers the present value of his income by $He^{-r(T-t)}$, so the consumer's response is

$$\widehat{c}_t = r\widehat{w}_t - \tau\left(He^{-r(T-t)}, \kappa\right)$$

Hence, the consumer will not think about the tax increase H when $He^{-r(T-t)} \leq \kappa$. Call $s \in [0, T)$ the first moment when he thinks about them (if it exists, i.e. if $H > \kappa$), otherwise we set $s = T$.

The next Proposition details the dynamics.

Proposition 12.10 (Myopic behavior and failure of Ricardian equivalence) *Suppose that taxes will go up at time T. While a rational agent would cut consumption at time 0, a sparse agent cuts consumption later, at a time $s = \max\left(0, \min\left(T, \frac{1}{r}\ln\frac{\kappa}{He^{-rT}}\right)\right)$. His consumption path is*

$$\widehat{c}_t = \begin{cases} 0 & \text{for } t < s \\ -He^{-r(T-s)} + \kappa\left(1 - r\left(t - s\right)\right) & \text{for } s \leq t < T \\ r\widehat{w}_T - H & \text{for } t \geq T \end{cases}$$

with $\widehat{w}_T = \frac{H}{r}\left(1 - e^{-t(T-s)}\right) - \kappa\left(T - s\right)$.

Let us take an example illustrated in Figure 6, with $r = 5\%$, $G = 2\%$, $T = 10$ years. This figure plots the change in consumption and wealth for the rational actor $\kappa = 0$ (black, solid) and progressively less rational agents: $\kappa = 0.01$ (blue, dotted), $\kappa = 0.025$ (red, dashed-dotted), $\kappa = 0.1$ (green, dashed). The traditional Ricardian consumer ($\kappa = 0$) immediately decreases his consumption by 2%, which leads to wealth accumulation until time T. In contrast, the BR consumer ($\kappa = 0.1$) doesn't react at all until $T = 10$ (hence he doesn't accumulated any wealth), and then cuts a lot of consumption. The value $\kappa = 0.01$ and $\kappa = 0.025$ display an intermediary behavior. For $\kappa = 0.025$, the consumer initially doesn't pay attention to the future tax. However, at a time $s = 4.5$ years (i.e., when there are 3.6 years remaining until the taxes are effective), he starts paying attention and starts savings for the future taxes. As the tax looms larger, the agent saves more. As the agent delayed his savings, he ends up cutting down on consumption more drastically when taxes are in effect.

[76] If taxes are collected later, then to guarantee the same present value, they need to be larger by a factor e^{rT}.

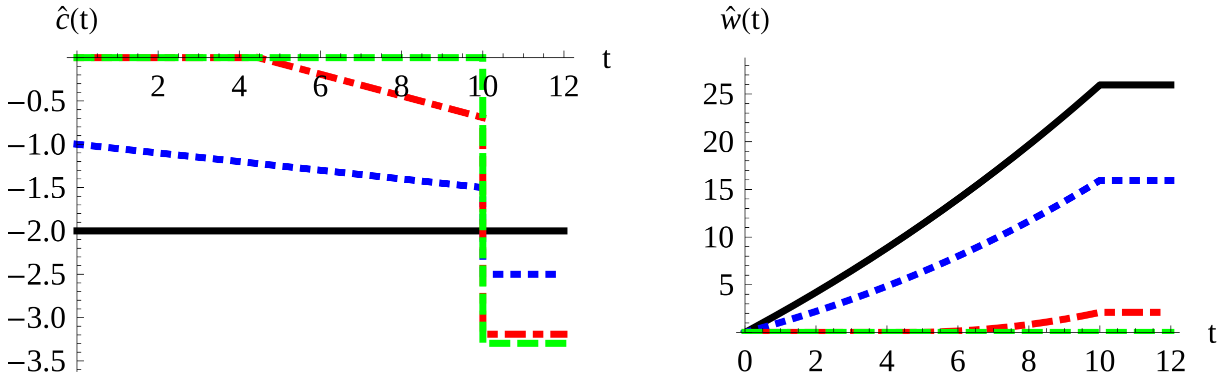

Figure 6: Reaction of consumption and wealth to an increase of future taxes, for different level of κ. *Notes.* At time 0, it is announced that taxes will be paid start at time $T = 10$. This Figure plots the change in consumption and wealth. The solid line is the prediction of the rational model (i.e.$\kappa = 0$), the other lines the reaction for different value of κ ($\kappa = 0.01$ (blue, dotted), $\kappa = 0.025$ (red, dashed-dotted), $\kappa = .1$ (green, dashed)). The very BR agents does not react at first, but starts reacting when he is closer to T. He reacts even more when taxes are in effect. As he delayed his savings, he needs to cut more on consumption when taxes start. Units are percentage points of previous steady state consumption. The amount is $G = 2\%$ of permanent income.

Smaller taxes generate a more delayed reaction. Controlling for the PV of taxes, consumers are better off with early rather than delayed taxes (as this allows them to smooth more).

Proof of Proposition 12.10 Taxes lower the present value of his income by $He^{-r(T-t)}$, so the consumer's response is:

$$\widehat{c}_t = r\widehat{w}_t - \tau\left(He^{-r(T-t)}, \kappa\right)$$

so wealth accumulation is: $\frac{d}{dt}\widehat{w}_t = r\widehat{w}_t - \widehat{c}_t = \tau\left(He^{-r(T-t)}, \kappa\right)$. The consumer starts thinking about it at a time s s.t. $He^{-r(T-s)} = \kappa$ (assuming that the solution is in $(0, T)$), i.e.

$$s = \max\left(0, \min\left(T, \frac{1}{r}\ln\frac{\kappa}{He^{-rT}}\right)\right) \tag{105}$$

First, consider the case $s < T$.

Then, for $t \in [s, T)$,

$$\begin{aligned}
\frac{d}{dt}\widehat{w}_t &= \tau\left(He^{-r(T-t)}, \kappa\right) = He^{-r(T-t)} - \kappa \\
\widehat{w}_t &= \int_s^t \left(He^{-r(T-t')} - \kappa\right) dt' \\
&= \frac{H}{r}e^{-rT}\left(e^{rt} - e^{rs}\right) - \kappa\,(t - s)
\end{aligned}$$

$$\widehat{c}_t = r\widehat{w}_t - \tau\left(He^{-r(T-t)}, \kappa\right) = r\left(\frac{H}{r}e^{-rT}\left(e^{rt} - e^{rs}\right) - \kappa\left(t - s\right)\right) - \left(He^{-r(T-t)} - \kappa\right)$$

$$\widehat{c}_t = -He^{-r(T-s)} + \kappa\left(1 - r\left(t - s\right)\right) \tag{106}$$

So at $t = T$

$$\widehat{w}_T = \frac{H}{r}\left(1 - e^{-r(T-s)}\right) - \kappa\left(T - t\right)$$

At T, the tax H is enacted, so that for $t \geq T$, the agent is aware of it. This yields

$$\begin{aligned} \widehat{c}_t &= r\widehat{w}_t - H \\ \frac{d}{dt}\widehat{w}_t &= r\widehat{w}_t - H - \widehat{c}_t \ = \text{investment income - taxes - consumption change} \\ &= 0 \end{aligned}$$

hence for $t > T$, $\widehat{w}_t = \widehat{w}_T$, and $\widehat{c}_t = r\widehat{w}_T - H$.

We conclude that consumption is

$$\widehat{c}_t = \begin{cases} 0 & \text{for } t < s \\ -He^{-r(T-s)} + \kappa\left(1 - r\left(t - s\right)\right) & \text{for } s \leq t < T \\ r\widehat{w}_T - H & \text{for } t \geq T \end{cases}$$

and wealth is

$$\widehat{w}_t = \begin{cases} 0 & \text{for } t < s \\ \frac{H}{r}e^{-rT}\left(e^{rt} - e^{rs}\right) - \kappa\left(t - s\right) & \text{for } s \leq t \leq T \\ \frac{H}{r}\left(1 - e^{-t(T-s)}\right) - \kappa\left(T - s\right) = \widehat{w}_T & \text{for } t \geq T \end{cases}$$

12.7 Active decision: Consumption or Savings?

Here we assume that the active decision was one of consumption. One could imagine that it would be in savings. Does this matter? First, for many variables, it does not matter: the impact of interest rates, future taxes, future income shocks etc. are the same whether a sparse agent uses the consumption frame or saving frame. However, the frame does matter for one variable: current income. Indeed, take the permanent-income setup. [77]

[77] Recall that $\widehat{c}^r_t = \frac{r}{r+\phi}\widehat{y}_t$, so

$$\widehat{c}^s_t = \frac{r}{r+\phi}m\widehat{y}_t \text{ under the consumption frame}$$

However, if the consumer choose savings, S_t, and then consumes $c_t = wy_t - S_t$, the rational amount is $\widehat{S}^r_t = \widehat{y}_t - \widehat{c}^r_t$, i.e. $\widehat{S}^r_t = \frac{\phi}{r+\phi}\widehat{y}_t$. Hence, the savings of a sparse agent is $\widehat{S}^s_t = \frac{\phi}{r+\phi}m\widehat{y}_t$, and the deviation of consumption is: $\widehat{c}^s_t = \widehat{y}_t - \widehat{S}^s_t$, Which frame does the agent use? One might posit that the agent takes the frame that yields the higher expected utility. To analyze this, we note the following result.

Proposition 12.11 (Welfare under the consumption vs savings frame) *The consumption frame yields greater utility than the savings frame if and only if $\phi_y > r$, i.e. if income shocks mean-revert faster than the interest rate.*

When $\phi_y > r$ (which is probably the relevant case), the "consumption" frame is indeed better for the agent. The reason is that consumption should be smooth, while savings could be bumpy as they absorb transitory income shocks. When the agent chooses consumption in an inattentive manner, it makes consumption automatically rather smooth. However, if the agent chooses savings inattentively, he makes savings smooth, but consumption needs to absorb the shocks, hence is quite volatile. Therefore, generally, to keep consumption smooth, choosing consumption inattentively is better than choosing savings inattentively. However, when income shocks are a random walk ($\phi_y = 0$), the savings frame is better. An inattentive agent will keep a constant savings and let consumption react one for one to income shock, which is the normatively correct behavior when income shocks are completely persistent.

Proof of Proposition 12.11 We use the content[78] and notations of Proposition 15.5. We set $x_t = \widehat{y}_t$. We have $F^w(w,x,c) = rw + x_t - c_t$ and $F^x(w,x) = -\phi x$.

Under the *consumption frame*, $a_t = c_t$, and $F^w_{aa} = 0$, so by Proposition 15.5, noting $\left[V^\delta_{xx}\right]^C$ the value of $V^\delta_{xx}(w,0)$ under the consumption frame:

$$\left[V^\delta_{xx}\right]^C = \frac{u''(c)}{r+2\phi_y}\left(c^s_y - c^r_y\right)^2 \tag{107}$$

and as $c^s_y = mc^r_y$ with $c^r_y = \frac{r}{r+\phi}$,

$$\left[V^\delta_{xx}\right]^C = \frac{u''(c)}{r+2\phi}(1-m)^2\left(\frac{r}{r+\phi}\right)^2$$

and the expected losses are (with $\sigma^2_y = \mathbb{E}\left[\widehat{y}^2_t\right]$)

$$\begin{aligned} L^C &= \frac{-1}{2}\left[V^\delta_{xx}\right]^C\sigma^2_y = \frac{-1}{2}\frac{u''(c)\,\sigma^2_y}{r+2\phi}(1-m)^2\left(\frac{r}{r+\phi}\right)^2 \\ &= A(1-m)^2 r^2 \end{aligned}$$

---

i.e.

$$\widehat{c}^s_t = \left(1 - \frac{m\phi}{r+\phi}\right)\widehat{y}_t \text{ under the savings frame}$$

which is generally not the same as $\widehat{c}^s_t$ under the consumption frame.

[78] We could also draw on the results in Cochrane (1989), with a variety of adjustments. Proposition 15.5 extend Cochrane's results (derived for consumption) to general dynamic problems.

Under the savings frame, a_t is savings, so $F^w = a_t$, and $c_t = rw_t + x_t - a_t$. Hence:

$$\left[V_{xx}^{\delta}\right]^S = \frac{u''(c)}{r+2\phi}\left(S_y^s - S_y^r\right)^2$$

and as $S_y^s = mS_y^r$, with $S_y^r = 1 - c_y^r = \frac{\phi}{r+\phi}$,

$$\left[V_{xx}^{\delta}\right]^S = \frac{u''(c)}{r+2\phi}(1-m)^2\left(\frac{\phi}{r+\phi}\right)^2$$

and expected losses are

$$L^S = \frac{-1}{2}\left[V_{xx}^{\delta}\right]^S \sigma_y^2 = A(1-m)^2\phi^2$$

The consumption frame yields greater utility than the savings frame if and only if $\phi_y > r$.

Losses from a general variable x. Using the same reasoning, the losses from not paying attention to a variable x is

$$L^x = \frac{-u''(c)}{r+2\phi}\sigma_x^2\left(c_x^s - c_x^{rat}\right)^2 = \frac{-u''(c)}{r+2\phi}\sigma_x^2 c_x^2 (1-m_x)^2$$

We parametrize the losses by the "equivalent permanent tax" λ^x such that

$$L^x = \mathbb{E}\int_0^\infty e^{-\rho t}\left[u(c_t) - u(c_t(1-\lambda))\right]dt$$

Hence, using a Taylor expansions, $\lambda^x = \frac{L}{u'(c)c/r}$. This gives

$$\lambda^x = \frac{1}{2}\frac{-\frac{u''(c)}{u'(c)c/r}}{r+2\phi}\sigma_x^2 c_x^2 (1-m_x)^2$$

i.e., using $\gamma = \frac{-cu''(c)}{u'(c)}$,

$$\lambda^x = \frac{1}{2}\frac{r\gamma}{r+2\phi}\left[\frac{c_x\sigma_x}{c}(1-m_x)\right]^2 \tag{108}$$

Proposition 12.12 *The losses from paying only attention m_x to variable x, expressed in terms of an "equivalent proportional losses in consumption", λ^x are*

$$\lambda^x = \frac{1}{2}\frac{r\gamma}{r+2\phi_x}\left[\frac{c_x\sigma_x}{c}(1-m_x)\right]^2 \tag{109}$$

where σ_x is the standard deviation of x, and $c_x = \frac{\partial c}{\partial x}$.

The calibration gives

$$\lambda^r = (1-m_r)^2 \times 0.03\%, \qquad \lambda^y = (1-m_y)^2 \times 3.0\% \tag{110}$$

It may be useful to see the effect in a simpler context. Take a 3 period model with $\beta = R = 1$, and an income shock with persistence ρ: $\widehat{y}_t = \rho^{t-1}\varepsilon$ for $t = 0, 1, 2$, with ε a mean-0 shock. Normatively, that should induce the change $\widehat{c} = (\widehat{c}_t)_{t=0,1,2} = (1, 1, 1)\,\frac{1+\rho+\rho^2}{3}\varepsilon$ (indeed, the total value of income has increased by $(1 + \rho + \rho^2)\,\varepsilon$). Let us now consider a BR agent with $m = 0$. However, under the consumption frame, $\widehat{c}^C = \left(0, \frac{1}{2}, \frac{1}{2} + \rho + \rho^2\right)\varepsilon$ (as there is no reaction of c_0, so that time-1 wealth increases by $\widehat{w}_1 = \varepsilon$, of which half is consumed at time 1, so $\widehat{c}_1^C = \frac{\varepsilon}{2}$). Under the savings frame, we get $\widehat{c}^S = (1, \rho, \rho^2)\,\varepsilon$ (savings doesn't change, consumption absorbs all the shocks). It is easy to verify that for ρ small, the utility is higher under the consumption frame, while the opposite for large ρ.[79] Indeed, when $\rho = 0$, $\widehat{c}^C = \left(0, \frac{1}{2}, \frac{1}{2}\right)\varepsilon$ and $\widehat{c}^S = (1, 0, 0)\,\varepsilon$, so there is more smoothing under the consumption frame. Other the other hand, with $\rho = 1$, $\widehat{c}^C = \left(0, \frac{1}{2}, \frac{5}{2}\right)\varepsilon$ and $\widehat{c}^S = (1, 1, 1)\,\varepsilon$, and there is more smoothing under the savings frame.

12.8 Intertemporal elasticity of substitution: controversies about its value

For many finance applications (e.g., Bansal and Yaron 2004, Barro 2009, Gabaix 2012), a high intertemporal elasticity of substitution (IES, denoted $\psi = 1/\gamma$) is important ($\psi > 1$). However, micro studies point to an IES of less than 1 (e.g., Hall 1988). I show how this may be due to the way econometricians proceed, by fitting the Euler equation, which yields $\ln c_{t+1} - \ln c_t = \frac{\widehat{\psi}}{R} r_t +$ constant, where $\widehat{\psi}$ is the measured IES.

I apply the infinite-horizon framework of Section 4.2. If the consumer "under-reacts to the interest rate," the measured IES will be biased towards 0. Using the above model, we can more precisely calculate that if consumers are boundedly rational (in the sense laid out above), then the estimated IES will be: $\widehat{\psi} = \overline{r}\left(w_t/c_t^d - 1\right) - b_r^s R\,(R - \rho_R)$. This is a point prediction that goes beyond Chetty (2012)'s prediction of an interval bound. Hence we obtain:

Proposition 12.13 *An econometrician fitting an Euler equation will estimate a downwardly-biased IES (intertemporal elasticity of substitution) if the agent is sparse:*

$$\widehat{\psi} = \psi - R\,(R - \rho_R)\,(b_r^s - b_r) < \psi$$

where $\widehat{\psi}$ is the estimated IES, ψ the true IES and $b_r^s - b_r$ is the difference between the sparse agent's and the traditional agent's interest-rate sensitivity of consumption.

The above calibration yields Figure 7, which plots the measured IES $\widehat{\psi}$ if the consumer is sparse with sparsity cost κ. If $\kappa = 0$, the consumer is the traditional, frictionless rational agent. We see that as κ increases, the IES becomes more and more biased. Hence, inattention may explain why while macro-finance studies require a high IES, microeconomic studies find a low IES.[80]

[79] To the leading order, $\widehat{u} = \frac{1}{2}u''\left(c^d\right)E\sum_t \widehat{c}_t^2$, so $\widehat{u}^C = \frac{1}{2}u''\left(c^d\right)\sigma_\varepsilon^2\left(\frac{1}{4} + \left(\frac{1}{2} + \rho + \rho^2\right)^2\right)$ and $\widehat{u}^S = \frac{1}{2}u''\left(c^d\right)\sigma_\varepsilon^2\left(1 + \rho^2 + \rho^4\right)$. This yields $\widehat{u}^C \geq \widehat{u}^S$ iff $\rho < \rho^* \simeq 0.32$.

[80] This is in the spirit of Gabaix and Laibson (2002)'s analysis of the biases in the estimation of the coefficient of

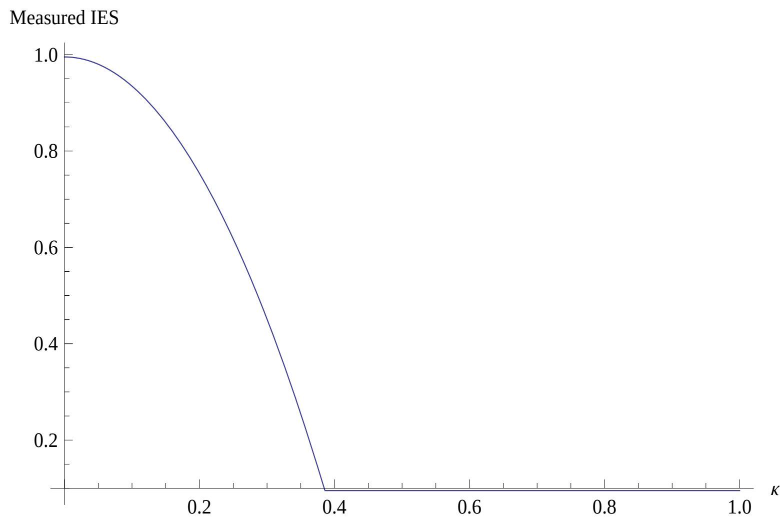

Figure 7: Measured intertemporal elasticitiy of substitution (IES), $\widehat{\psi}$, if the consumer is sparse with cost κ, while the econometrician assumes he is fully rational. The true IES is $\psi = 1$.

12.9 Source-dependent Marginal Propensity to Consume

The agent has initial wealth w and future income y, can consume c at time 1, and invest the savings at a rate R. Hence, the problem is as follows: given an initial wealth w, solve $\max_c V = u(c) + \mathbb{E}[v(y + R(w-c))]$, where income is $y = y_* + \sum_{i=1}^n y_i$ - there are n sources of income y_i with mean 0. Let us study the solution of this problem with the algorithm. The agent observes the income sources sparsely: he uses the model $y(m) = y_* + \sum_{i=1}^n m_i y_i$, with m_i to be determined. Applying this model, we obtain (assuming exponential utility with absolute risk aversion γ for simplicity)

Proposition 12.14 *Time-1 consumption is* $c = \frac{1}{1+R}(Rw + \delta/\gamma - \gamma\sigma_\varepsilon^2/2 + y_* + \sum_i m_i y_i)$, $m_i = \tau(1, \frac{\kappa^m \sigma_{c_2}}{\sigma_{y_i}})$. *The marginal propensity to consume (MPC) at time 1 out of income source* i *is*

$$MPC_i^s = MPC_i^r \cdot m_i, \tag{111}$$

where $MPC_i^s = (\partial c/\partial y_i)^s$ *is the MPC under the sparse model, and* $MPC_i^r = (\partial c/\partial y_i)^r$ *is the MPC under the traditional rational-actor model. Hence, in the sparse model, unlike in the traditional model, the marginal propensity to consume is source-dependent.*

Different income sources have different marginal propensities to consume – reminiscent of Thaler (1985)'s mental accounts. Equation (111) makes another prediction: consumers pay more attention to sources of income that usually have large consequences, i.e. have a high σ_{y_i}. Slightly extending the model, it is plausible that a shock to the stock market does not affect the agent's disposable

risk aversion with inattentive agents, in a different context and a more tractable model. See also Fuster, Laibson and Mendel (2010) for a model where agents' use of simplified models leads to departures from the standard aggregate model.

income much – hence, there will be little sensitivity to it. The MPC out of wage income will be higher than the MPC to consume out of portfolio income.

This model shares similarities with models of inattention based on a fixed cost of observing information. Those models are rich and relatively complex (they necessitate many periods, or either many agents or complex, non-linear boundaries for the multidimensional s, S rules, or signal extraction as in Sims 2003), whereas the present model is simpler and can be applied with one or several periods. As a result, the present model, with an equation like (111), lends itself more directly to empirical evaluation.

12.10 Cognitive Discounting of Future News

Suppose that the agent is told that he may receive \$7 in 3 periods, but that those \$7 may disappear, with a survival probability of ρ_y per period. Then, the \$7 should count at $\rho_y^3 7$. However, a sparse agent could replace ρ_y by $\rho_y(m_{y'}) = m_{y'}\rho_y + (1-m_{y'})\rho_y^d$. If $\rho_y(m_{y'}) < \rho_y$, he's thinking "Let me not bother with those potential future things", or "I'll believe it when I see it", or something akin to "a bird in the hand is worth two in the bush".

The framework easily accommodates that behavior of "cognitive discounting". Call f_t^y the vector of future flows, whose $s-$th component will arrive in s periods.[81] In the rational model, with no decay,

$$f_{t+1}^y = Lf_t + \varepsilon_{t+1}^{f^y}$$

where L is the left-shift operator $L(f_1, f_2, f_3, ...) = (f_2, f_3, ...)$. An innovation ε_{t+1}^f codifies announcement. For instance, in the initial example $\varepsilon_t^{f^y} = (0, 0, 7, 0, 0, ...)$. When there is cognitive discounting, operator L is replaced by $\rho_{y'}(m)L$.

To come back to our consumption problem, the problem is, under the subjective model (with $Z = (w, r_t, f_t^y, f_t^r)$):

$$\begin{aligned} F^w(Z,m) &= (R + m_r\hat{r}_t)(w_t - c_t) + \bar{y} + m_y\hat{y}_t \\ f_{t+1}^x &= m_{f^x}Lf_t^x + \varepsilon_{t+1}^x, \qquad \hat{x}_t = k_1^x \cdot f_t^x \text{ for } x = \hat{r}, \hat{y} \end{aligned}$$

Here, the value of m_{f^x} "dampens" the appreciation of future movements in variable x. Intuitively, because the future is harder to predict, its simulations are dampened.

Proposition 12.15 *In the cognitive discounting specification, the behavioral policy is:*

$$\hat{c}_t^s = \mathbb{E}_t\left[\sum_{\tau\geq t}\frac{1}{R^{\tau-t+1}}\left(m_r m_{r'}^{\tau-t} b_r(w_t)\hat{r}_\tau + m_y m_{y'}^{\tau-t} b_y\hat{y}_\tau\right)\right], \tag{112}$$

All those expressions hold up to second order terms.

[81] More generaelly, $E_t[y_{t+s}] = (f_t^y)_s = k_s^y \cdot f_t^y$, with $k_s^y = (0, ..., 0, 1, 0, 0, ...)$ the vector selecting the $s-$th component.

Formulation (112) encapsulates two different forms of inattention. First, the agent may not think about interest rate at all if $m_r = 0$. Second, he may discount future news, if $m_{r'} < 1$. Indeed, he discounts future news arriving in T periods by a factor $m_{r'}^T$. In addition, this discounting is source-specific: if news about future interest rates are less important than news about future income (something we will compute soon), they are (cognitively) discounted more.[82]

12.11 Extension of the basic 3-period example

Using the simplification function The value (39) is a bit complicated. This is where the simplification operator S (defined in Definition 15.2) intervenes. Applying it (with the same notations as in the motivating example before and after Definition 15.2), we obtain $V^{1,S} := S(V^1)$, i.e.

$$V^{1,S}(w,x) = 2u\left(\frac{w_1 + x}{2}\right) \tag{113}$$

The value is the same as V^1, up to $O(x^2)$ terms: $V^1(w_1,x) = V^{1,S}(w,x) + O(x^2)$. The attention-augmented value function at time 1 is

$$V^1\left(w,x,m^V\right) = m^V V^1(w,x) + \left(1-m^V\right) V^{1,S}(w,x)$$

At time 0, the agent does $\text{smax}_{c_0;m}\, v^0(c_0,x,m_0)$, with $m_0 = \left(m_0^x, m^V\right)$ and

$$v^0(c_0,w_0,x,m_0) := u(c_0) + V^1\left(w_0 - c_0, m_0^x x, m^V\right) \tag{114}$$

The FOC is $v_{c_0}^0 = 0$ with

$$v_{c_0}^0 = u'(c_0) - V_w^1\left(w_0 - c_0, m_0^x x, m^V\right).$$

We have $V_{w,m^V}^1 = 0$ at the default $m_0^d = (0,0)$, so $\frac{\partial c_0}{\partial m^V}_{|m^d=0} = 0$ and the optimal attention is $m^V = 0$: the agent uses the proxy value function, not the exactly rational one (we will see soon that attention m^V can be non-zero using the 2-step smax, but it is still likely to be 0 if κ is not too small).

We note that if $m^V > 0$, the FOC is more complex. The FOC is

$$u'(c_0) = \left(1-m^V\right) u'\left(\frac{w_1 + m_0 x}{2}\right) + m^V \frac{1}{2}\left[u'\left(\frac{w_1 + m_1 x}{2}\right) + u'\left(\frac{w_1 + (2-m_1)x}{2}\right)\right]$$

Still, to the first order, the decision is the same (as per Proposition 3.9). Making the problem simpler at every period, via the $m^V = 0$ device, makes the problem more tractable for both the

[82] Psychologically, why the decay? This may be because often "promises are not kept", or "something intervenes" (so that the agent anchors on a decay $\rho_y^d < 1$), or simply because things far in the future are generally less easy to perceive.

agent and the economist examining him. We next study this, using the 2-step sparse max.

Using the 2-step sparse max We have so far used the plain sparse max. This led to $m^V = 0$, the exclusive reliance on the simplified value function. We now calculate what happens when using the twice-iterated sparse max of Definition 15.1.

To endogenize m^V, we use the twice-iterated smax: $\text{smax}^2_{c;m}\, v^0(c_0, w_0, x, m)$ with $m = (m_0^x, m^V)$. At the first round, $v^0_{c_0, m_V} = 0$, so $m_0^V(1) = 0$, and, as before, $m_0^x(1) = \mathcal{A}\left(\frac{1}{6\kappa} u''\left(\frac{w_0}{3}\right)\sigma_x^2\right)$.

At the second round, now $m^d = (m_0^x(1), 0)$. The easy part is the attention to x, which is slightly different than at step 1:

$$m_0^x(1) = \mathcal{A}\left(\frac{1}{6\kappa} u''\left(\frac{w_0 + m_0^x(0)\, x}{3}\right)\sigma_x^2\right)$$

The more novel part is to calculate m_V. We have, with $w_1 = w_0 - c_0$ and calling $x^s := m_0^x x$,

$$\begin{aligned}
v^0_{c,m_V}\left(c_0, w_0, x, m_0^x, m^V\right) &= \partial_c\left[V^1(w_0 - c_0, x^s) - V^1\left(w_0 - c_0, x^s, m^V = 0\right)\right] \\
&= -\frac{1}{2}u'\left(\frac{w_1 + m_1 x^s}{2}\right) - \frac{1}{2}u'\left(\frac{w_1 + (2 - m_1)\, x^s}{2}\right) + u'\left(\frac{w_1 + x^s}{2}\right)
\end{aligned}$$

Doing a Taylor expansion of the consumptions $\frac{w_1 + m_1 x^s}{2}$ and $\frac{w_1 + (2-m_1)x^s}{2}$ around their mean

$$c^d = \frac{w_1 + x^s}{2} = \frac{w_1 + m_0^x x}{2}$$

we obtain

$$\begin{aligned}
v^0_{c,m_V} &= -\frac{1}{2}u'\left(c^d + (m_1 - 1)\frac{x^s}{2}\right) - \frac{1}{2}u'\left(c^d - (m_1 - 1)\frac{x^s}{2}\right) + u'\left(c^d\right) \\
&= -\frac{1}{2}u'''\left(c^d\right)(m_1 - 1)^2\left(\frac{x^s}{2}\right)^2 \times 2 + o\left(x^2\right) \\
&= -\frac{1}{4}u'''\left(c^d\right)(m_1 - 1)^2\left(m_0^x x\right)^2 + o\left(x^2\right)
\end{aligned}$$

Likewise, $v^0_{cc|m=m_0^d(1)} = \frac{3}{2}u''\left(c^d\right)$. So, the impact of m_V is

$$\frac{\partial c_0}{\partial m_V} = -\frac{v^0_{c,m_V}}{v^0_{cc}} = -\frac{1}{6}\frac{u'''\left(c^d\right)}{u''\left(c^d\right)}(m_1 - 1)^2\left(m_0^x x\right)^2 + o\left(x^2\right)$$

Hence, for a small x, the attention m^V to the difference between the difference between the true and proxy value functions (i.e., $V^1\left(w_1, x, m^V\right)$ for $m^V = 1$ vs $m^v = 0$) is:

$$\begin{aligned} m_0^V &= A\left(\frac{1}{\kappa}\mathbb{E}\left[\left(\frac{\partial c_0}{\partial m_V}\right)^2 v_{cc}^0\right]\right) \\ &= A\left(\frac{1}{\kappa}\mathbb{E}\left[\left(\frac{1}{6}\frac{u'''\left(c^d\right)}{u''\left(c^d\right)}\left(m_1-1\right)^2\left(m_0^x\right)^2 x^2\right)^2\right]\frac{3}{2}u''\left(c^d\right)\right) \\ &= A\left(\frac{1}{24\kappa}\left(\frac{u'''\left(c^d\right)}{u''\left(c^d\right)}\right)^2\left(m_1-1\right)^4\left(m_0^x\right)^4\mathbb{E}\left[x^4\right]u''\left(c^d\right)\right) \end{aligned} \tag{115}$$

It is instructive to take the limit of small κ using a sparsity-inducing cost function ($g'(0) > 0$). To have $m_0^x > 0$, we need $\frac{\sigma_x^2}{\kappa}$ large enough, so $\sigma_x \succeq \kappa^{1/2}$. To have $m_0^V > 0$, we need $\frac{\sigma_x^4}{\kappa}$ large enough, i.e. $\sigma_x \succeq \kappa^{1/4}$, which is a much higher hurdle ($\frac{\kappa^{1/4}}{\kappa^{1/2}} \to \infty$) for small κ. We formalize this.

Proposition 12.16 (Attention to a variable, vs attention to the fine properties of how the value function depends on that variable) *Suppose a succession of problems (indexed by κ going to 0) such that there are positive constants B, B', ε such that for κ small enough: $B\kappa^{1/2-\varepsilon} \le \sigma_x(\kappa) \le B'\kappa^{1/4+\varepsilon}$. Then, the agent will have $m_0^x > 0$ and $m_0^V = 0$ when κ is small enough. This is, the agent pays attention to the disturbance x, but not to the subtle difference between the true and proxy value functions (i.e., $V^1\left(w_1, x, m^V\right)$ for $m^V = 1$ vs $m^V = 0$).*

In plain terms: because thinking about the nuances m^V in $V\left(x, m^V\right)$, one needs to think about x at all. Hence, in many situations, we have $m^V = 0$ and $m^x > 0$. Indeed, we cannot have, with just one state, variable $m^x = 0$ and $m^V > 0$.

In particular, for our 3-period problem for κ small enough but not too small, $m^V = 0$ and $m_0^x > 0$ - the agent uses the simplified value function, but still pays attention to x, like in the basic smax case. This is one reason it is useful to use the basic smax: it gets to the essence of the more complex patterns that can later be refined using the iterated smax.

13 Complements to the life-cycle model

Here I record variants on the life-cycle models of Section 2. What happens with various attention functions, in discrete and continuous time, etc. This is useful to get a "feel" for the model in a concrete, substantial setting.

13.1 Derivation in continuous time

It is instructive to do the proof in continuous time, using the notations of Section 2. The budget constraint is $\dot{w}_t := \frac{dw_t}{dt} = y_t - c_t$, where a dot denotes a time-derivative.

If the agent is rational, his value function is

$$V^r(w_t, \widehat{y}, t) = (T-t)\, u\left(\bar{y} + \frac{w_t + x}{T-t}\right)$$

Indeed, the optimal policy is to smooth consumption over the time $[t, T]$, exhausting final resources, which are $w_t + (T-t)\,\bar{y} + x$, so that $c_t = \frac{w_t + (T-t)\bar{y} + x}{T-t}$, and the value function is $V^r = (T-t)\, u(c_t)$. I take $V^p = V^r$ for the proxy value function.

Hence, at time t, the Bellman equation is

$$\operatorname*{smax}_{c;m} v(c_t, m, \widehat{y}, t) \text{ with } v(c_t, m, \widehat{y}, t) := u(c_t)\, dt + V^r(w_t - c_t dt, m\widehat{y}, t)\,.$$

The f.o.c. is:

$$\begin{aligned} v_c &= u'(c_t)\, dt - V^r_w(w_t - c_t dt, m\widehat{y}, t)\, dt = \left[u'(c_t) - V^r_w(w_t, m\widehat{y}, t)\right] dt + O\left(dt^2\right) \\ v_{cc} &= u''(c_t)\, dt + V^r_w(w_t - c_t dt, m\widehat{y}, t)\,(dt)^2 = u''(c_t)\, dt + O\left(dt^2\right) \end{aligned}$$

Hence, the optimal policy is given by $v_c = 0$, i.e. $u'(c_t) = V^r_w(w_t, m\widehat{y}, t) = u'\left(\bar{y} + \frac{w_t + mx}{T-t}\right)$, so

$$c_t = \bar{y} + \frac{w_t + mx}{T-t} = c^d_t + m b_t,$$

where

$$b_t := \frac{x}{T-t}$$

is the normative sensitivity response to $\widehat{y}$.

Next, for the allocation of attention, we form

$$\max_m \frac{-1}{2} v_{cc_{|m=0}} \left(\frac{\partial c_t}{\partial m}\right)^2 (m_t - 1)^2 - \kappa_t g(m)\, dt$$

i.e. the attention is

$$m^*_t = \mathcal{A}\left(\frac{\left|u''\left(c^d_t\right)\right| b^2_t}{\kappa}\right)$$

Using the definition of the truncation function $\tau(b, k) := b\mathcal{A}\left(\frac{b^2}{k^2}\right)$ (equation 12) the response is

$$c_t - c^d_t = m^*_t b_t = \tau\left(b_t, \sqrt{\frac{\kappa_t}{\left|u''\left(c^d_t\right)\right|}}\right) \tag{116}$$

13.2 Calculation of the consumption path

The basic result (Proposition 2.1) gives the consumption policy, which is enough to simulate the consumption path, say with a computer. Here I explore specifications to obtain an analytic representation. As always, continuous time is much simpler to derive and cleaner than discrete time. Still, because it is more elementary, I start with discrete time.

13.2.1 Consumption Path in Discrete Time

We will see that the path is as follows. Consider

$$B(s) := \frac{-x}{T-s} + \frac{\bar{\kappa}^2}{x}(T-s-1) \tag{117}$$

which is related to marginal net benefit of thinking.

Proposition 13.1 (Basic life-cycle: consumption path in discrete time) *Call s the first time at which the agent thinks about retirement, $s = \inf\{s \in [0, L] : B(s) \geq 0\}$. Hence, the agent can think of retirement at time $s = 0$ (this is the case if $B(0) \geq 0$), at time a later time before retirement (this is the case if $B(0) < 0 < B(L)$, and s is the solution of $B(s) = 0$ if that is an integer). Or he may never think about it until actual retirement, $s = L$.*

Before he thinks of retirement ($t < s$), consumption is $c_t = \bar{y} + \frac{w_0}{T}$ and $w_t = \left(1 - \frac{t}{T}\right) w_0$. After the agent thinks of retirement ($t \in [s, L)$):

$$c_t = \frac{w_0}{T} + \bar{y} + \frac{2\kappa^2}{x}(t-s) - B(s)$$

and wealth is $w_t = \left(1 - \frac{t}{T}\right) w_0 - \frac{\kappa^2}{x}(t-s)(t-s-1) + B(s)(t-s)$ for $t \in [s, L]$.

Consumption after retirement (for $t \in [L, T)$) is constant, at $c_t = \frac{w_L}{T-K} + \bar{y} + \hat{y}$.

We next derive this. We follow Section 2, which gives:

$$c_t = \frac{w_t + m_t x}{T-t} + y$$

We suppose the scaling: $\kappa = \bar{\kappa}^2 u''\left(c_t^d\right)$, given $v'' = u''\left(1 + \frac{1}{T-t-1}\right)$,

$$\begin{aligned} m_t &= \mathcal{A}\left(\frac{v_{cc} c_m^2}{\kappa}\right) = \mathcal{A}\left(\frac{v''}{u''}\frac{c_m^2}{\bar{\kappa}^2}\right) = \mathcal{A}\left(\left(1 + \frac{1}{T-t-1}\right)\left(\frac{x}{T-t}\right)^2\right) \\ &= \mathcal{A}\left(\frac{1}{(T-t-1)(T-t)}\frac{x^2}{\bar{\kappa}^2}\right) \end{aligned}$$

Using $\mathcal{A}(y) = \max\left(1 - \frac{1}{|y|}, 0\right)$,

$$m_t \frac{x}{T-t} = \frac{x}{T-t}\left(1 - \frac{\bar{\kappa}^2 (T-t-1)(T-t)}{x^2}\right) = \frac{x}{T-t} - \frac{\bar{\kappa}^2 (T-t-1)}{x}$$
$$= -B(s)$$

where $B(s)$ was defined in (117). Note that $B(s)$ is increasing for $s < T$.

To concentrate on the key difficulty, I take the case $w_0 = 0$.

If the solution is interior ($s \in (0,T)$), the agent thinks about retirement for s such that

$$B(s) = 0 \tag{118}$$

Hence, when attention is positive, we have:

$$c_t = y + \frac{w_t + x}{T-t} - \frac{\kappa^2 (T-t-1)}{x} \tag{119}$$

We need to calculate the case where the agent saves before retirement. We look for a solution of the type, for $t \in [s, L)$:

$$c_t = y + A(t-s) + b$$

for some constants A, b. At $t = s$, (119) implies $b = -B(s)$. For time $t \in [s, L)$, we have:

$$w_t = -\sum_{t'=s}^{t-1} (c_{t'} - y) = -\frac{A}{2}(t-s)(t-s-1) + B(s)(t-s)$$

We want to verify (119), which we will express: $g(t) = 0$ with

$$\begin{aligned}
g(t) &:= (T-t)\left(c_t - y - \frac{w_t + x}{T-t} + \frac{\kappa^2 (T-t-1)}{x}\right) \\
&= (T-t)(c_t - y) - w_t - x + (T-t)(T-t-1)\frac{\kappa^2}{x} \\
&= (T-t)(A(t-s) - B(s)) + \frac{A}{2}(t-s)(t-s-1) + B(s)(t-s) - x + (T-t)(T-t-1)\frac{\kappa^2}{x}
\end{aligned} \tag{120}$$

This is a polynomial in t. The coefficient of t^2 must be 0, so $-\frac{A}{2} + \frac{\kappa^2}{x} = 0$, and

$$A = \frac{2\kappa^2}{x}$$

To make sure that $g(t) = 0$, we check first $g(s) = 0$. But

$$g(s) = -(T-s)\,B(s) - x + \frac{\bar{\kappa}^2 (T-s)(T-s-1)}{x} = 0$$

because of (117). Finally,

$$g(T) = \frac{\kappa^2}{x}(T-s)(T-s-1) - B(s)(T-s) - x = 0$$

again. Given g is a polynomial of degree at most 1, with $g(s) = g(T) = 0$, g is identically 0.

13.2.2 Consumption Path in Continous Time

Using the scaling $\kappa_t = \bar{\kappa}^2 \left|u''\left(c_t^d\right)\right|$. I set: $\kappa_t = \bar{\kappa}^2 \left|u''\left(c_t^d\right)\right|$. The consumption policy before retirement is

$$c_t = \frac{w_t}{T-t} + \tau\left(\frac{x}{T-t}, \kappa\right) + y$$

The agent will not think about the change in income when $-\frac{x}{T-t} < \kappa$. Call $s \in [0, L)$ the first moment when he thinks about it. If the solution is interior, $\frac{x}{T-s} = \kappa$, i.e. $s = T + \frac{x}{\kappa}$. In general, we need to windsorize at 0 and L :

$$s := \max\left(0, \min\left(L, T + \frac{x}{\kappa}\right)\right) \tag{121}$$

Note that this is for $\hat{y} \le 0$. For $\hat{y} > 0$, we have $s := \max\left(0, \min\left(L, T - \frac{x}{\kappa}\right)\right)$.

For $t \le L$, define the deviations from the policy that doesn't pay attention to retirement x:

$$\begin{aligned}
\hat{c}_t &= c_t - \frac{w_0}{T} - y \\
\hat{w}_t &= \int_0^t -\hat{c}_t\, dt = w_t - \frac{T-t}{T} w_0
\end{aligned}$$

We plug $\hat{c}_t$ and $\hat{w}_t$ into the optimal consumption policy to get

$$\hat{c}_t = \frac{\hat{w}_t}{T-t} + \tau\left(\frac{x}{T-t}, \kappa\right)$$

First case, using truncation function $\tau_1(b, \kappa) = b \max\left(1 - \frac{\kappa^2}{b^2}, 0\right)$. I first assume that $s < L$. For $t \in [s, L)$ the agent's response is

$$\begin{aligned}
\hat{c}_t &= \frac{\hat{w}_t}{T-t} + \tau\left(\frac{x}{T-t}, \kappa\right) = \frac{\hat{w}_t}{T-t} + \frac{x}{T-t}\left(1 - \frac{\kappa^2}{\left(\frac{x}{T-t}\right)^2}\right) \\
&= \frac{\hat{w}_t + x}{T-t} + \frac{\kappa^2}{x}(T-t)
\end{aligned}$$

Take derivative with respect to t, this yields, using $\frac{d\hat{w}_t}{dt} = -\hat{c}_t$,

$$\begin{aligned}\frac{d\hat{c}_t}{dt} &= \frac{-\hat{c}_t}{T-t} + \frac{\hat{w}_t + x}{(T-t)^2} - \frac{\kappa^2}{x} \\ &= \frac{-\hat{c}_t}{T-t} + \frac{\hat{c}_t - \frac{\kappa^2}{x}(T-t)}{(T-t)} - \frac{\kappa^2}{x} \\ &= \frac{-2\kappa^2}{x}\end{aligned}$$

with boundary condition that $\hat{w}_s = 0$. Solving for $\hat{c}_t$, for $t \in [s, L]$

$$\hat{c}_t = \frac{-2\kappa^2}{x}(t-s)$$

The wealth at time t is

$$\hat{w}_t = \int_s^t -\hat{c_{t'}}\, dt' = \frac{\kappa^2}{x}(t-s)^2$$

Second case. Now use truncation function $\tau_{L_1}(b, \kappa) = sign(b) \max(|b| - |\kappa|, 0)$. After time s, the agent's response is

$$\hat{c}_t = \frac{\hat{w}_t}{T-t} + \tau\left(\frac{x}{T-t}, \kappa\right) = \frac{\int_s^t -\hat{c}_t\, dt}{T-t} - \left(-\frac{x}{T-t} - \kappa\right)$$

Taking derivative with respect to t yields

$$\begin{aligned}\frac{d\hat{c}_t}{dt} &= \frac{-\hat{c}_t}{T-t} + \frac{\hat{w}_t}{(T-t)^2} + \frac{x}{(T-t)^2} \\ &= \frac{-\hat{c}_t}{T-t} + \frac{\hat{c}_t - \frac{x}{T-t} - \kappa}{(T-t)} + \frac{x}{(T-t)^2} \\ &= \frac{-\kappa}{T-t}\end{aligned}$$

with boundary condition that $\hat{c}_s = 0$. Can solve $\hat{c}_t$ as,

$$\hat{c}_t = \kappa \ln\left(\frac{T-t}{T-s}\right)$$

The wealth at time $t \leq L$ will be

$$\hat{w}_t = \int_s^t -\hat{c}_t\, dt = \kappa(T-s)\left(\frac{T-t}{T-s}\ln\frac{T-t}{T-s} - \frac{T-t}{T-s} + 1\right)$$

Case 1: With $\tau_1(b,\kappa) = b\max\left(1-\frac{\kappa^2}{b^2},0\right)$, his consumption path is

$$c_t = \begin{cases} \frac{w_0}{T}+y & \text{for } t<s \\ \frac{w_0}{T}+y-2\frac{\kappa^2}{x}\frac{t-s}{T-s} & \text{for } s\le t<L \\ \frac{w_L}{T-L}+y+\hat{y} & \text{for } t\ge L \end{cases}$$

with $w_L = \hat{w}_L + \frac{B}{T}w_0 = \kappa\frac{(L-s)^2}{T-s}+\frac{B}{T}w_0$.

Case 2: With $\tau_{L_1}(b,\kappa) = sign(b)\max(|b|-|\kappa|,0)$, his consumption path is

$$c_t = \begin{cases} \frac{w_0}{T}+y & \text{for } t<s \\ \kappa\ln\left(\frac{T-t}{T-s}\right)+\frac{w_0}{T}+y & \text{for } s\le t<L \\ \frac{w_L}{T-L}+y+\hat{y} & \text{for } t\ge L \end{cases}$$

with $w_L = \hat{w}_L + \frac{B}{T}w_0 = \kappa(T-s)\left(\frac{B}{T-s}\ln\frac{B}{T-s}-\frac{B}{T-s}+1\right)+\frac{B}{T}w_0$.

Using a different scaling, $\kappa_t = \bar{\kappa}^2\left|\left(c_t^d\right)^2 u''\left(c_t^d\right)\right|$ Here I explore a different scaling $\kappa_t = \bar{\kappa}^2\left|\left(c_t^d\right)^2 u''\left(c_t^d\right)\right|$, with $c_t^d = \bar{y}+\frac{w_t}{T-t}$. This renders $\bar{\kappa}$ dimensionless and potentially portable from one situation to the next. Then (116) gives:

$$c_t - c_t^d = \tau\left(b_t, \sqrt{\frac{\kappa_t}{\left|u''\left(c_t^d\right)\right|}}\right) = \tau\left(b_t, \bar{\kappa}c_t^d\right)$$

The agent is all set – he just follows that policy. Now let's turn to the economist's role, to trace out the implications of that policy.

I use the truncation function $\tau_{L_1}(b,\kappa) = sign(b)\max(|b|-|\kappa|,0)$, because it yields simpler calculations. I assuming $\hat{y}<0$, to focus on the retirement case. We have, when $b_t := \frac{x}{T-t} < -\bar{\kappa}c_t^d$, $\tau\left(b_t,\bar{\kappa}c_t^d\right) = b_t+\bar{\kappa}c_t^d$, so

$$\begin{aligned} \hat{c}_t &:= c_t-\bar{y} = \left(c_t^d-\bar{y}\right)+\left(c_t-c_t^d\right) \\ &= \frac{\hat{w}_t}{T-t}+b_t+\bar{\kappa}c_t^d = \frac{\hat{w}_t}{T-t}+\frac{x}{T-t}+\bar{\kappa}\left(\bar{y}+\frac{\hat{w}_t}{T-t}\right) \end{aligned}$$

i.e.

$$\widehat{c}_t = \frac{(1+\bar{\kappa})\hat{w}_t+x}{T-t}+\bar{\kappa}\bar{y} \tag{122}$$

Let us examine the first time s at which the agent thinks about retirement. It if it in $(0,L)$, we have $\hat{w}_s = 0$, and $\frac{x}{T-t} + \bar{\kappa}\bar{y} = 0$, so $s = T + \frac{x}{\bar{\kappa}\bar{y}}$. In general, we windsorize by 0 and L, and obtain:

$$s := \max\left(0, \min\left(L, T + \frac{x}{\bar{\kappa}\bar{y}}\right)\right) \tag{123}$$

To find c_s, we again apply (122), observing that $\hat{w}_s = 0$:

$$\begin{aligned} c_s &= \bar{y} + \frac{x}{T} + \bar{\kappa}\bar{y} \text{ if } s = 0 \\ &= \bar{y} \text{ if } s \in (0, L) \end{aligned} \tag{124}$$

From then on, we assume $s < L$.

After the agent has starting thinking about retirement, we have (122), so taking the time derivative, using $\widehat{\dot{w}}_t = -\hat{c}_t$, we have:

$$\begin{aligned} \dot{\widehat{c}}_t &= \frac{(1+\bar{\kappa})\,\hat{w}_t + x}{(T-t)^2} + \frac{(1+\bar{\kappa})}{T-t}\widehat{\dot{w}}_t = \frac{\widehat{c}_t - \bar{\kappa}\bar{y}}{T-t} - \frac{(1+\bar{\kappa})\,\widehat{c}_t}{T-t} \\ &= -\bar{\kappa}\frac{\widehat{c}_t + \bar{y}}{T-t} \end{aligned}$$

As $c_t = \widehat{c}_t + \bar{y}$, we have $\dot{c}_t = -\kappa\frac{X_t}{T-t}$, so $\frac{d\ln c_t}{dt} = \frac{-\kappa}{T-t}$, $\ln c_t = \kappa \ln(T-t) + a$, and

$$c_t = A\,(T-t)^{\kappa}$$

for a constant A. Given the value at c_s

$$c_t = \left(\frac{T-t}{T-s}\right)^{\kappa} c_s. \tag{125}$$

The wealth at time $t \leq L$ is

$$\hat{w}_t = \int_s^t -(c_t - \bar{y})\,dt = (t-s)\,\bar{y} - \frac{1}{1+\kappa}\left((T-s) - \frac{(T-t)^{1+\kappa}}{(T-s)^{\kappa}}\right) c_s. \tag{126}$$

13.3 Lifecycle: Policy of the hyperbolic log agent

The log case is particularly clean (Barro 1999). The agent at time t has decision utility: $\ln c_t + \beta\sum_{\tau=t+1}^{T-1} \ln c_\tau$. Given full wealth (including discounted future income y_t) call Ω_t, his policy will be some $c_t = \mu_t\Omega_t$, for μ_t independent of Ω_t, so we shall have

$$\sum_{\tau=t}^{T-1} \ln c_\tau = (T-t)\ln\Omega_t + K_t$$

for some constant K_t. The implies a consumption policy

$$c_t = \arg\max u(c_t) + \beta(T-t-1)\ln(\Omega_t - c_t) + K_{t+1}$$

The first order condition is

$$\frac{1}{c_t} = \frac{\beta(T-t-1)}{\Omega_t - c_t}$$

i.e.

$$c_t = \frac{\Omega_t}{1+\beta(T-t-1)}$$

We see that there is no "cliff" at retirement. The agent makes no distinction between pre and post retirement income.

14 Proofs Omitted in the Paper

Proof of Lemma 4.2 *Exact result.* The problem is

$$\max_{(c_t)_{t\geq 0}} \sum_{t\geq 0} \beta_t \frac{c_t^{1-\gamma}}{1-\gamma} \text{ s.t. } \sum_{t\geq 0} q_t c_t \leq \Omega_0$$

where $q_t = \frac{1}{(1+r_0)...(1+r_{t-1})}$ is the time-0 Arrow-Debreu price of a dollar received at t, and β_t is the discount factor (which is not necessarily of the form β^t here), and

$$\Omega_0 := w_0 + \sum_{t>0} q_t y_t$$

is the full wealth. Forming the Lagrangian,

$$L = \sum_{t\geq 0} \beta_t \frac{c_t^{1-\gamma}}{1-\gamma} + \lambda\left(\Omega_0 - \sum_{t\geq 0} q_t c_t\right)$$

we have $\beta_t c_t^{-\gamma} = \lambda q_t$, i.e. (with $\psi = \frac{1}{\gamma}$), $c_t = c_0\left(\frac{\beta_t}{q_t}\right)^{\psi}$ for some c_0. The budget constraint gives:

$$\Omega_0 = \sum_{t\geq 0} q_t c_t = c_0 \sum_{t\geq 0} \beta_t^{\psi} q_t^{1-\psi}$$

i.e.

$$c_0 = \mu\Omega_0, \qquad \mu := \frac{1}{\sum_{t\geq 0}\beta_t^{\psi} q_t^{1-\psi}} \tag{127}$$

Given $V'(\Omega_0) = u'(c_0) = u'(\mu\Omega_0)$, we have (as the function is also homogeneous of degree $1-\gamma$): $V(\Omega_0) = \frac{1}{\mu}u(\mu\Omega_0)$.

Suppose now that $\beta_t = \beta^t$ and $\beta R = 1$. Then the interest rate is constant, $q_t = R^{-t}$, and $\mu = \frac{1}{\sum_{t\geq 0} R^{-t}} = \frac{1}{\frac{1}{1-\frac{1}{R}}} = \frac{\bar{r}}{R}$ and $\Omega_0 = w_0 + \frac{\bar{y}}{r}$. So, $c_0 = \mu\Omega_0 = \frac{rw_0+\bar{y}}{R}$.

Taylor expansion, first in the deterministic case. The impact of a change dy_τ is very easy: $dc_0 = \mu d\Omega_0 = \frac{\bar{r}}{R}\frac{dy_\tau}{R^{\tau+1}}$. The impact of an interest rate is more delicate. Consider a change change dr_τ, for just one date τ. It creates a bond price change $dq_t = \frac{-1}{R^{t+1}} dr_\tau 1_{t>\tau}$, so that

$$\sum_{t\geq 0} dq_t = \sum_{t\geq 0} \frac{-1}{R^{t+1}} dr_\tau 1_{t>\tau} = \sum_{t\geq \tau+1} \frac{-1}{R^{t+1}} dr_\tau = \frac{-1}{rR^{\tau+1}} dr_\tau$$

This gives

$$\begin{aligned}\frac{d\mu}{\mu} &= -\mu\left(1-\psi\right)\sum_{t\geq 0}\beta_t^{\psi} q_t^{-\psi} dq_t = -\frac{\bar{r}}{R}\left(1-\psi\right)\sum_{t\geq 0} dq_t \\ &= \left(1-\psi\right)\frac{\bar{r}}{R}\frac{1}{rR^{\tau+1}} dr_\tau = \mu\left(1-\psi\right)\frac{dr_\tau}{R^{\tau+2}}\end{aligned}$$

Also, $d\Omega_0 = \bar{y}\sum_{t\geq 1} dq_t = \frac{-\bar{y}}{r}\frac{dr_\tau}{R^{\tau+1}}$. Recalling that $c_0 = \mu\Omega_0$:

$$\begin{aligned}dc_0 &= \mu\Omega_0\frac{d\mu}{\mu} + \mu d\Omega_0 = c_0\left(1-\psi\right)\frac{dr_\tau}{R^{\tau+2}} + \frac{\bar{r}}{R}\frac{-\bar{y}}{r}\frac{dr_\tau}{R^{\tau+1}} \\ &= \left(-\psi c_0 + \frac{rw_0+\bar{y}}{R} - \bar{y}\right)\frac{dr_\tau}{R^{\tau+2}} = \left(-\psi c_0 + \frac{r\left(w_0-\bar{y}\right)}{R}\right)\frac{dr_\tau}{R^{\tau+2}}\end{aligned}$$

This gives announced value.

Stochastic case. As we are reasoning up to $O\left(\|x\|^2\right)$, we can take the certainty equivalent, e.g. use $\mathbb{E}\left[f\left(x\right)\right] = f\left(\mathbb{E}\left[x\right]\right) + O\left(var\left(x\right)\right)$ for a C^2 function f. Section 14) provides the (standard) details.

Value function. Write $c_t = \mu_t\left(w_t - \bar{y}\right) + \nu_t$, for values μ_t, ν_t independent of w_t. Because one dollar now can be consumed today, we have $V'\left(w_t\right) = u'\left(c_t\right)$, and indeed $V\left(w_t\right) = \frac{1}{\mu_t}u\left(\mu_t\left(w_t-\bar{y}\right)+\nu_t\right)$. Using (45), in particular $c_t^d = \frac{r\left(w_t-\bar{y}\right)}{R} + \bar{y}$ and $b_r\left(w_t\right) := \frac{\frac{\bar{r}}{R}\left(w_t-\bar{y}\right)-\psi c^d}{R}$, we obtain:

$$\begin{aligned}\mu_t w_t + \nu_t = c_t = c_t^d + \hat{c}_t &= c_t^d + \mathbb{E}_t\left[\sum_{\tau\geq t}\frac{1}{R^{\tau-t+1}}\left(b_r\left(w_t\right)\hat{r}_\tau + b_y\hat{y}_\tau\right)\right] \\ &= \frac{\bar{r}\left(w_t-\bar{y}\right)}{R} + \bar{y} + \mathbb{E}_t\left[\sum_{\tau\geq t}\frac{1}{R^{\tau-t+1}}\left(\frac{\frac{\bar{r}}{R}\left(w_t-\bar{y}\right) - \psi\left(\frac{r\left(w_t-\bar{y}\right)}{R}+\bar{y}\right)}{R}\hat{r}_\tau + b_y\hat{y}_\tau\right)\right]\end{aligned}$$

which gives the announced values.

Proof of Lemma 4.4 In the AR(1) case,

$$\mathbb{E}_t\left[\sum_{\tau\geq t}\frac{1}{R^{\tau-t+1}}\hat{r}_\tau\right]=\sum_{\tau\geq t}\frac{1}{R^{\tau-t+1}}\rho_r^{\tau-t}\hat{r}_t=\sum_{\tau\geq t}\frac{1}{R}\left(\frac{\rho_r}{R}\right)^{\tau-t}\hat{r}_t=\frac{1}{1-\frac{\rho_r}{R}}\frac{\hat{r}_t}{R}=\frac{1}{R-\rho_r}\hat{r}_t$$

$$\mu_t=\frac{\bar{r}}{R}+(1-\psi)\frac{\bar{r}}{R}\mathbb{E}_t[\sum_{\tau\geq t}\frac{\hat{r}_\tau}{R^{\tau-t+2}}]=\frac{\bar{r}}{R}+(1-\psi)\frac{\bar{r}}{R^2}\frac{1}{R-\rho_r}\hat{r}_t$$

$$\nu_t=\bar{y}+\mathbb{E}_t[\sum_{\tau\geq t}\frac{\frac{-\psi\bar{y}}{R}\hat{r}_\tau+b_y\hat{y}_\tau}{R^{\tau-t+1}}]=\bar{y}+\frac{-\psi\bar{y}}{R}\frac{1}{R-\rho_r}\hat{r}_t+b_y\frac{1}{R-\rho_y}\hat{y}_t$$

Proof of Lemma 3.6 This is a variant on the standard proof. We have

$$\mathcal{T}(V,V^p)(Z)-\mathcal{T}\left(\tilde{V},V^p\right)(Z)=\beta\mathbb{E}\left[V(Z')-\tilde{V}(Z')\right] \tag{128}$$

with $Z':=F(z,a(Z,V^p),\mu)$.

For (i), using $\left|V(Z')-\tilde{V}(Z')\right|\leq\left\|V-\tilde{V}\right\|_\infty$,

$$\left|\mathcal{T}(V,V^p)(Z)-\mathcal{T}\left(\tilde{V},V^p\right)(Z)\right|\leq\beta\mathbb{E}\left[\left|V(Z')-\tilde{V}(Z')\right|\right]\leq\beta\mathbb{E}\left[\left\|V-\tilde{V}\right\|_\infty\right]=\beta\left\|V-\tilde{V}\right\|_\infty$$

and taking the sup on the left-hand side,

$$\left\|\mathcal{T}(V,V^p)-\mathcal{T}\left(\tilde{V},V^p\right)\right\|_\infty\leq\beta\left\|V-\tilde{V}\right\|_\infty.$$

For (ii), if $V(Z')-\tilde{V}(Z')\leq 0$ for all Z', then (128) implies that $\mathcal{T}(V,V^p)(Z)-\mathcal{T}\left(\tilde{V},V^p\right)(Z)\leq 0$. The operator is monotone.

Proof of Lemma 3.6 This is the usual fixed point argument. Define $V_0:=V^p$, and for $n\geq 0$, $V_{n+1}=\mathcal{T}(V_n,V^p)$. By Lemma 3.6,

$$\|V_{n+1}-V_n\|_\infty=\|\mathcal{T}(V_n,V^p)-\mathcal{T}(V_{n-1},V^p)\|_\infty\leq\beta\|V_n-V_{n-1}\|_\infty$$

hence V_n is a Cauchy sequence and converges in a complete metric space.

Proof of Proposition 3.9 Proposition 3.11 implies that $V(w,x)=V^r(w,x)+O\left(\|x\|^2\right)$.[83] Next, Lemma 3.12 implies that the optimal policy satisfies $a(w,x)=a^r(w,x)+O\left(\|x\|^2\right)$. Next, decompose $a^r(w,x)=a^d(w)+\sum_i b_i(w)x_i+O\left(\|x\|^2\right)$. Then, using the policy V without sparse

[83] Proposition 3.9 is stated in the text before some results this proof uses (this is useful to make the flow of the paper more natural), but there is no logical circularity.

max, we have

$$a(w,x) = a^d(w) + \sum_i b_i(w) x_i + O\left(\|x\|^2\right)$$

Finally, using Lemma 3.3, we have:

$$a^s(w,x) = a^d(w) + \sum_i \tau\left(b_i(w), \frac{\kappa_a}{\sigma_i}\right) x_i + O\left(\|x\|^2\right).$$

Proof of Proposition 3.11 It is in Section 16.1.

Proof of Lemma 3.12 The reasoning is a direct transposition of the arguments in the proof of Proposition 3.11.

Proof of Lemma 3.13 Given a value function $V(w,x)$, Lemma 10.2 shows that, up to $O\left(\|x\|^2\right)$ terms, $a(w,x)$ just depend on $V(w,0), V_w(w,0)$ (but not on $V_{xx}(w,0)$). The Lemma assumes that the two functions $V^p(w,x) = V^{p'}(w,x)$ have the same values of $V(w,0), V_x(w,0)$. Hence, their actions $a(w,x)$ are the same up to $O\left(\|x\|^2\right)$.

Proof of Proposition 3.14 We will prove by induction on $q \geq 0$ that the following property holds $H_q : V^{(q)}(z) = V^r(z) + O\left(\|x\|^2\right)$.

This is true by assumption for $q = 0$. Suppose H_q holds, we will see that H_{q+1} holds. By Lemma 3.13,

$$a\left(z, V^{(q)}\right) = a\left(z, V^{(0)}\right) + O\left(\|x\|^2\right) \tag{129}$$

Because $a^{(0)}(z)$ is C^1, we also have $a\left(z, V^{(0)}\right) = a^r(z) + O(\|x\|)$, so $a\left(z, V^{(q)}\right) = a^r(z) + O(\|x\|)$. Lemma 3.12 in turns implies that $V^{(q+1)}(z) = V^r(z) + O\left(\|x\|^2\right)$.

Proof of Lemma 4.2: Complements The following completes the arguments given in the paper.

Stochastic case. With stochastic r_τ, y_τ, we use $\mathbb{E}[f(X)] = f(\mathbb{E}[X]) + O(var(X))$ when the random variable X has small variance. Technically, we assume a bounded distribution of X, and f is C^2 in a compact interval I containing $\mathbb{E}[X]$ in its interior.[84] This, way, we can move from the deterministic version of (45) to its expectation, capturing absorbing the uncertainty terms in the $O\left(\|x\|^2\right)$.

[84] The proof is standard. Normalizing $EX = 0$, and calling x here just a real number, we use $f(x) = f(0) + f'(0)x + R(x)$ with $R(x) = \int_0^x f''(t)(x-t)\,dt$. So $f(X) = f(0) + f'(0)X + R(X)$, and

$$Ef(X) = f(EX) + ER(X)$$

Using$|R(x)| \leq Cx^2$ for $C := \frac{1}{2}\sup_{x\in I}|f''(x)|$, we have $|Ef(X) - f(EX)| \leq CE\left[X^2\right] = Cvar(X)$. So, $Ef(X) = f(EX) + O(var(X))$.

Proof of Lemma 4.4 One proof is that it is just a straightforward corollary of (51). Here I show another proof, via a Taylor expansion of the default value function.

The Bellman equation is

$$V(w,r) = \max_c u(c) + \beta V((R+r)(w-c) + y', r') \tag{130a}$$

I suppress the expectation operator, as the shocks are assumed to be small. We assume a law of motion:

$$r' = \rho r + \varepsilon'$$

Call next-period wealth w':

$$w' = (R+r)(w-c) + y'$$

We assume that the agent knows the simple model where the interest rate is always at its average, $r \equiv 0$. As is well-known, the optimal policy is $c = rw + y$, and, with $R = 1 + \overline{r}$,

$$V(w) = A\left(w + w^H\right)^{1-\gamma} / (1-\gamma),\ w^H = Y/\overline{r}, A = (\overline{r}/R)^{-\gamma}$$

First, we differentiate the Bellman equation with respect to the new variable:

$$\begin{aligned} V_r(w,r) &= \beta V'_{w'}(w',r')\frac{\partial w'}{\partial r} + \beta V'_{r'}(w',r')\frac{\partial r'}{\partial r} \\ V_r(w,r) &= \beta V'_{w'}(w',r')(w-c) + \beta V'_{r'}(w',r')\rho \end{aligned} \tag{131}$$

Evaluating at $r = 0$, this leads to

$$V_r(w,0) = V^d_w(w)\frac{\beta(w-c)}{1-\beta\rho}$$

We now take the total derivative with respect to w , $D_w V = \partial_w V + \frac{da}{dw}\partial_a V$, e.g. the full impact of a change in w, including the impact it has on a change in the consumption c. The baseline policy is $c(w) = \overline{r}w/R + \overline{y}$, so $D_w c = \overline{r} : R$, and $D_w w' = d(R(w-c))/dw = R - R\overline{r}/R = 1$.

$$\begin{aligned} D_w c &= \overline{r}/R \\ D_w w' &= 1 \end{aligned}$$

This means that one extra dollar of wealth received today translates into exactly one dollar of wealth next period; its interest income, r, is entirely consumed.

So differentiate (using the total derivative) equation 131. We obtain

$$\begin{aligned}
\beta^{-1}V_{wr}(w,r) &= V'_{w'w'}(w',r')(D_{w'}w)\cdot(w-c)+V'_{w'}(w',r')D_w(w-c)+V'_{w'r'}(w',r')\rho D_w w' \\
&= V'_{w'w'}(w',r')(w-c)+V'_{w'}(w',r')(1-\frac{\overline{r}}{R})+V'_{w'r'}(w',r')\rho
\end{aligned}$$

so, using

$$V'_{w'w'}(w',r') = -\gamma V'_w \cdot \frac{1}{w+w^H} = -\gamma V'_w \cdot \frac{\overline{r}}{Rc}$$

$$V_{w,r} = \frac{\beta \frac{V'_{w'}}{R}\left(1-\gamma\overline{r}\left(\frac{w-c}{c}\right)\right)}{1-\rho\beta}$$

Finally, let's derive the impact of a change in r on c. We have

$$V_w = \beta(R+r)V'_{w'} = u'(c)$$

so

$$\begin{aligned}
\frac{dc}{dr} &= \frac{V_{wr}}{u''(c)} = \frac{-1}{u''(c)}\frac{V_w}{R}\frac{1-\gamma\overline{r}\left(\frac{w}{c}-1\right)}{R-\rho_r\beta} \\
&= \frac{-1}{\gamma u'(c)\,c}\frac{V_w}{R}\frac{1-\gamma\overline{r}\left(\frac{w}{c}-1\right)}{R-\rho_r} \\
\frac{dc}{c} &= \frac{1}{R}\frac{\overline{r}\left(\frac{w}{c}-1\right)-1/\gamma}{R-\rho_r}dr = b^r dr \\
b_r &= \frac{\overline{r}\left(w_t-c^d\right)-\frac{c^d}{\gamma}}{R(R-\rho_r)} \\
&= \frac{\overline{r}\left(w_t-\frac{rw_t+y}{R}\right)-\frac{c^d}{\gamma}}{R(R-\rho_r)} = \frac{\frac{\overline{r}}{R}(w_t-y)-\frac{c^d}{\gamma}}{R(R-\rho_r)}
\end{aligned}$$

We note that the result

$$b_y = \frac{\overline{r}}{R(R-\rho_y)}, \qquad b_r = \frac{\overline{r}\left(w_t-c^d\right)-\frac{c^d}{\gamma}}{R(R-\rho_r)}$$

becomes, in continuous time:

$$b_y = \frac{\overline{r}}{\overline{r}+\phi_y}, \qquad b_r = \frac{\overline{r}w_t-\psi c_t^d}{\overline{r}+\phi_r} \tag{132}$$

Proof of Proposition 5.2 When $\phi>0$, we saw that

$$\phi = \left(\frac{\xi}{r+\phi}-\overline{\kappa}\left(r+\frac{\xi}{r+\phi}\right)^2\frac{r+\phi}{\xi}\right)$$

Let $\psi := \dfrac{r+\phi}{\xi} \neq 0$. Then

$$\phi = \psi^{-1} - \kappa(r + \psi^{-1})^2\psi,$$

which is equivalent to

$$\begin{aligned}\psi(\xi\psi - r) = \psi\phi &= 1 - \kappa[(r + \psi^{-1})\psi]^2 \\ &= 1 - \kappa(r\psi + 1)^2 \\ &= 1 - \kappa(r^2\psi^2 + 2r\psi + 1).\end{aligned}$$

Rearranging yields

$$(\xi + \kappa r^2)\psi^2 + (2\kappa - 1)r\psi + (\kappa - 1) = 0.$$

The quadratic formula then gives

$$\psi = \frac{(1-2\kappa)r \pm \sqrt{\Delta}}{2(\xi + \kappa r^2)},$$

where

$$\begin{aligned}\Delta &= [(2\kappa - 1)r]^2 - 4(\xi + \kappa r^2)(\kappa - 1) \\ &= r^2\left[(2\kappa - 1)^2 - 4\kappa(\kappa - 1)\right] + 4\xi(1-\kappa) \\ &= r^2\left[(4\kappa^2 - 4\kappa + 1) - (4\kappa^2 - 4\kappa)\right] + 4\xi(1-\kappa) \\ &= r^2 + 4\xi(1-\kappa).\end{aligned}$$

In the case $\kappa = 0$, the correct root is the higher one for ψ (i.e., it's the higher root of $\phi = \frac{\xi}{r+\phi}$, the one with the $+\sqrt{\Delta}$ sign). Hence, $\psi = \dfrac{(1-2\kappa)r + \sqrt{\Delta}}{2(\xi + \kappa r^2)}$. Finally,

$$\begin{aligned}\phi = \xi\psi - r &= \frac{\xi\left[(1-2\kappa)r + \sqrt{\Delta}\right] - 2(\xi + \kappa r^2)r}{2(\xi + \kappa r^2)} \\ &= \frac{\left[\xi(1-2\kappa) - 2(\xi + \kappa r^2)\right]r + \xi\sqrt{\Delta}}{2(\xi + \kappa r^2)} = \frac{-\left[2\kappa r^2 + 2\xi\kappa + \xi\right]r + \xi\sqrt{\Delta}}{2(\xi + \kappa r^2)} \\ &= \frac{-\left[2\kappa r^2 + 2\xi\kappa + \xi\right]r + \xi\sqrt{r^2 + 4\xi(1-\kappa)}}{2(\xi + \kappa r^2)}\end{aligned}$$

15 Extensions of the Basic Model

15.1 Evaluating the benefits of thinking at the true model

In the basic sparse max, benefits of thinking are evaluated at the default model. Here is a simple extension where they're evaluated at the true model. This is useful to avoid "starvation" in some extreme examples.

Call

$$\omega_i(m) := \mathbb{E}\left[-a'_{m_i}\left(m^d, x\right) u_{aa}\left(a\left(m^d, x\right), m, x\right) a_{m_i}\left(m^d, x\right)\right]$$

the prospective benefits of thinking, evaluated at model m. The basic sparse max sets attention to

$$m_i^* = \mathcal{A}\left(\frac{1}{\kappa}\omega_i\left(m^d\right)\right)$$

But one could enrich it e.g. as

$$m_i^* = \mathcal{A}\left(\frac{1}{\kappa}\max\left(\omega_i\left(m^d\right), \frac{\omega_i(\mu)}{K}\right)\right)$$

with $K > 1$. The max features two term: in the first one ($\omega_i\left(m^d\right)$), the benefits are evaluated at the default model; in the second term ($\omega_i(\mu)$) benefits are evaluated at the true model. To capture the fact that this is a more complex procedure, a penalty of $K > 1$ is applied, for some K, e.g. $K = 10$.

A benefit is that then the model "detects the danger of starvation". In the 3-period model, we have

$$m_1^* = \mathcal{A}\left(\frac{\sigma_x^2}{2\kappa}\max\left\{\left|u''\left(\frac{w_1}{2}\right)\right|, \frac{1}{K}\left|u''\left(\frac{w_1}{2}+x\right)\right|\right\}\right)$$

so that if $\frac{w_1}{2}+x$ is too close to a starvation level, then the second part is "active", and attention becomes higher (if $u''' > 0$). For instance, if $\left|u''\left(\frac{w_1}{2}+x\right)\right| = \infty$, then $m_1^* = 1$, and the consumer becomes fully attentive.

Likewise, we'll have

$$m_0^* = \mathcal{A}\left(\frac{\sigma_x^2}{6\kappa}\max\left\{\left|u''\left(\frac{w_0}{3}\right)\right|, \frac{1}{K}\left|u''\left(\frac{\frac{2w_0}{3}+x}{2}\right)\right|\right\}\right)$$

as the value function is evaluated as a derivative of $V^{1,p}(w_1, x) = 2u\left(\frac{w_1+x}{2}\right)$.

15.2 Finite-difference in the sensitivity to m

When we calculate $a_{m_i} = \frac{\partial a}{\partial m_i}$ in Definition 3.1, the following variant $\overline{\partial}_{m_i} a$ is sometimes useful. We first need to define the finite-difference operator:

$$\Delta_{m_i} g(m) := g(m)_{m_i=1,\ m_{-i}=0} - g(0) = g(0, ..., 0, 1, 0, ..., 0) - g(0)$$

where the 1 is at the i-th coordinate of m. This is the "finite difference" analogue to $\partial_{m_i} g(m) = \frac{\partial g}{\partial m_i}_{|m=0}$. Next, we define:

$$\overline{\partial}_{m_i} a(m, x) := \Delta_{m_i}(a_x \cdot x) = \Delta_{m_i}\left(\sum_k \left(\frac{\partial}{\partial x_k} a(m, x)_{|x=0}\right) x_k\right) = (\Delta_{m_i}\partial_x)(a) \cdot x$$

Note that if $a(x, m) = \sum b_i m_i x_i$, then $\overline{\partial}_{m_i} a(m, x) = b_i m_i = \partial_{m_i} a(m, x)$. However, the definition using $\overline{\partial}_{m_i}$ generalizes better. For instance, if m is one-dimensional $(m = m_1)$ and

$$a(m_1, x) = \sum_{i=1}^{3} m_1^i b_i x_i$$

then $\overline{\partial}_{m_1} a(m, x) = \sum_{i=1}^{3} b_i x_i$, whereas $\partial_{m_1} a(m, x) = \frac{\partial a(x)}{\partial m_1}_{|m=0} = b_1 x_1$. The higher-power terms m^2, m^3 are "invisible" when using ∂_{m_1}, but "visible" when using $\overline{\partial}_{m_1}$.

15.3 Taking into account the costs of thinking in the value function

One could take into account the costs of thinking in the value function. This will complicate the issues a bit and change the optimal action only by second order terms. Therefore, it's best not to do that in the first model.

Should thinking costs be taken into account?

There are some reasons to do it. If attention is a resource, then its cost should be taken into account.

There are also reasons *not* to do it. First, we're modelling a BR agent and imagining that the BR agent optimizing today will take into account future thinking costs may assume too much rationality. Technically when optimizing, the agent may take a default value of 0 for κ^V. Second, it could be that the costs in his decision utility (the ones used when deciding) are not the actual costs of thinking. This is the case if the agent misoptimizes on inattention, i.e. does as if the cost was κ – but perhaps the true cost is $\kappa^V = 10\kappa$ or $\kappa^V = 0.1\kappa$.

In the basic statement of the model, I opted for the simplest version of the framework. Here is an expanded version that does take them into account.

The selection of the action is still (27). Calling $m^*(Z, V^p)$ the attention return by the smax in

(27), and the value function iteration in (29) becomes

$$\mathcal{T}(V,V^p)(Z) := u(a(Z,V^p), Z, \mu) - \kappa^V G(m^*(Z,V^p)) + \beta \mathbb{E}[V(F(z, a(Z,V^p), \mu))]$$

where $G(m) = \sum_i g(m_i)$, and κ^V is the perceived cost as included in the value function, and the state vector Z is expanded to include κ^V as a state variable. The basic sparse max corresponds to taking $\kappa^V = 0$ at the Bellman iteration stage.

How do results change? First, the basic smax agent does not change his action at all. What changes is for the $q-$iterated agent case with $q \geq 1$. Second, in accordance of Section 3.5, this change makes only a second order difference in the action.

Formally, this can be interpreted by enriching the action space to $\bar{a} = (a, m)$ and having attention be part of the action vector and an expanded utility function:

$$\bar{u}(a, m, Z, \mu) = u(a, Z, \mu) - \kappa^V G(m)$$

Then, the perceived decision utility may or may not capture the correct value of κ. This is close to the perspective taken in Farhi and Gabaix (2015, Section 6.1).

15.4 Iterated Static Sparse Max

In some cases, it is useful to have a generalization of the basic sparse max.

Definition 15.1 (Iterated sparse max for static problems) *The $K-$times iterated sparse max, $\text{smax}^K_{a;m|m^d} u(a,x)$, is defined by the following procedure. Define $m^d(k)_{k=1}$ to be the initial default attention, m^d.*

Start at round $k = 1$. At each round $k \leq K$, apply the regular smax, using the default $m^d(k)$: $\text{smax}_{a;m|m^d(k)} u(a,x,m)$, and call $m^*(k)$ and $a^*(k)$ the resulting attention. Define then $m^d(k+1) = m^*(k)$.

Stop at the end of round $k = K$, and return $m^*(K)$ and $a^*(K)$, the optimal attention and action at the last iteration.

Illustration. Suppose that $u(a,x) = -\frac{1}{2}(a - x_1(1+x_2))^2$ so that the rational policy is $a^r(x_1,x_2) = x_1(1+x_2)$. If the agent doesn't think of x_1 (replacing it with $x_1 = 0$), then he should not think about x_2.

We next apply the iterated smax outlined in Definition 15.1, iterating twice ($K = 2$). Initial default attention is $m^d(1) = (0,0)$. We start at step $k = 1$. We observe that so $a^r_{x_1} = 1 + x_2$, $a^r_{x_2} = x_1$, which gives

$$m_1^*(1) = \mathcal{A}\left(\frac{\sigma_1^2}{\kappa}\right), \qquad m_2^*(1) = 0$$

So, at the beginning of the second step, the default is $m^d(2) = m^*(1)$. Again applying the plain smax but with that default $m^d(2)$, we have

$$m_1^*(2) = \mathcal{A}\left(\frac{\sigma_1^2}{\kappa}\right), \qquad m_2^*(2) = \mathcal{A}\left(\frac{m_1^*(1)^2 \sigma_2^2}{\kappa}\right)$$

Hence, the action is $a = a^r(m^*(2) \odot x) = m_1^*(2) x_1 (1 + m_2^*(2) x_2)$. We also see that, as $\kappa \to 0$, the action converges to the rational action.[85]

15.5 Proportional Thinking

Here is a simple microfoundation for the scale-free κ of Section 10.2.2, equation (133). Thinking about m_i implies some "mental costs" $g(m_i)$. These costs translate into some trembling in the action, so that, with $a^{**} = \arg\max_a v(a, x, m^*)$, the actual action is:

$$a = a^* + \tilde{\eta}$$

where $\tilde{\eta}$ is a mean 0 noise with standard deviation:

$$Stdev(\eta) = \sqrt{2}\bar{\kappa} a^{\natural} \left(\sum_i g(m_i)\right)^{1/2}$$

The size of the noise proportional to the typical scale $a^{\natural}$ of the action (this proportional is encountered in much of psychophysics, e.g. in the Weber-Fechner law)[86], and increasing in mental activity $m.$. We call $\sqrt{2}\bar{\kappa}$ the factor of proportionality.

Hence, the utility losses from this noise $L = -\mathbb{E}[v(a^* + \eta, x) - v(a^*, x)]$ are, to the leading order:

$$L = -\frac{1}{2}\mathbb{E}\left[\eta^2\right] v_{aa} = -v_{aa}\bar{\kappa}^2 \left(a^{\natural}\right)^2 \sum_i g(m_i)$$

Hence, as in Gabaix (2014, Lemma 2), the utility losses from imperfect inattention are, to the leading order:

$$\frac{1}{2}\sum_{i,j=1\ldots n}(1 - m_i)\Lambda_{ij}(1 - m_j) + \kappa \sum_i g(m_i)$$

[85] This iterated smax suffices for the problems considered in this paper. For other purposes, one could imagine a variant where the default is at say $m^d = (\varepsilon, ..., \varepsilon)$, for some $\varepsilon > 0$, so as to better "probe" the importance of all variables.

[86] The microfoundation of that is probably that noisy computations are made in the brain in a scale-free space, and then at the end multiplied by $a^{\natural}$ to get an action to scale. This generates the proportionality to $a^{\natural}$. This type of thinking, however, it still speculative at this stage (Glimcher 2010).

with $\Lambda_{ij} := -\mathbb{E}\left[a_{m_i} V_{aa} a_{m_j}\right]$ and

$$\kappa = \left(\bar{\kappa} a^{\natural}\right)^2 \left|v_{aa}\left(a^d\left(x, m^d\right)\right)\right|. \tag{133}$$

15.6 Simplification of Functions

15.6.1 Taylor expansion inside a value function

We develop here a bit of simple machinery to reflect how the agent can "simplify" a function (in practice a value function), by forcing them to have a given functional form.

A motivating example. Suppose that the agent consumes $c_1 = \frac{w}{2} + y_1$ and $c_2 = \frac{w}{2} + y_2$, where $y = (y_1, y_2)$ can be viewed as small. His rational value function, assuming no discounting, is

$$v(y) = u\left(\frac{w}{2} + y_1\right) + u\left(\frac{w}{2} + y_2\right)$$

The agent may wish to use a simplified representation of this function. We observe that $v(y) = v^S(y) + O\left(\|y\|^2\right)$ with

$$v^S(y) := 2u\left(\frac{w + y_1 + y_2}{2}\right)$$

We shall take this function V^S as a "simplified" representation of v. We can then form a more general function: $v\left(y, m^V\right) := \left(1 - m^V\right) v^S(y) + m^V v(y)$. If $m^V = 1$, then the agent uses the rational value function. If $m^V = 0$, then the agent uses the proxy value function v^S, which is in some sense simpler.

The following definition generalizes that thought and codifies the creation of a "simplified" value function.

Definition 15.2 (Simplifying function) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that $f_{x_i}(x)_{|x=0} \neq 0$ for all i, and $\phi : \{1, ...p\} \to \{1, ...n\}$. Call $\mathcal{E}^f := \{v \in C^1(\mathbb{R}^p, \mathbb{R})$ such that $v(0) = f(0)\}$. We define the simplification function $S_{f,\phi} : \mathcal{E}^f \to \mathcal{E}^f$ by*

$$\left(S_{f,\phi}(v)\right)(y) := f(b \cdot y) \tag{134}$$

where b is the uniquely determined matrix $b \in \mathbb{R}^{n \times p}$ such $b_{ij} = 0$ unless $i = \phi(j)$ and

$$v(y) = f(b \cdot y) + o(\|y\|) \tag{135}$$

Furthermore, $b_{ij} = \frac{v_{y_j}(y)_{|y=0}}{f_{x_i}(x)_{|x=0}}$ if $i = \phi(j)$, $b_{ij}(x) = 0$ otherwise.

We prove the b is indeed unique.

Proof We want $v(y) = f(b \cdot y) + o(\|y\|)$. This is equivalent to

$$v_{y_j}(y)_{|y=0} = \sum_i f_i b_{ij} = f_{\phi(j)} b_{\phi(i)j}.$$

Inspecting the Taylor expansions gives the result. □

This defines an attention-augmented function

$$v\left(y, m^V\right) := \left(1 - m^V\right) v^S(y) + m^V v(y)$$

where parameter m^V captures the attention to the true value v.

Basically $f(b \cdot y)$ is like a non-linear Taylor expansion of $v(y)$. For instance, in our introductory example, $f(x) = 2u\left(\frac{w+x}{2}\right)$, $y = (y_1, y_2)$, $n = 1$, $p = 2$, $\phi(j) = 1$, and $b = (1, 1)$.

Here are two other variants of the same idea. Suppose that we have a stochastic variable and a variant of the Black-Scholes model, with stochastic volatility. Then, we may approximate the value function in by tweaking the implied volatility: $V(x_t, S, K, r, t) = V^{\mathrm{BS}}(\bar{\sigma} + ax_t + o(x_t), S, K, r, t)$, where V^{BS} is the regular Black-Scholes formula, so that the simplified function is

$$V^S(x_t, S, K, r, t) = V^{\mathrm{BS}}(\bar{\sigma} + ax_t, S, K, r, t)$$

Suppose that the agent estimates a distribution, $h(y)$, where y are parameters of the distribution. The agent may wish to replace this distribution by a distribution with a simpler functional form, say a Gaussian: then f is a Gaussian distribution approximating the distribution h, perhaps by matching h's mean and variance.

15.6.2 Just paying attention to first order terms

Suppose that the problem is:

$$\max_a u(a, x)$$

which gives $a = a^d + b \cdot x + O\left(\|x\|^2\right)$, with $b = -u_{aa}^{-1} u_{ax}$. Define $u^d(a) := u(a, 0)$, and $a^d = \arg\max_a u(a, 0)$. Suppose we have an agent that actually does:

$$\tilde{a} = a^d + b \cdot x$$

i.e. exactly discards the second order terms. How to we represent that agent?

First, we could define a "Taylor sparse max", that given a problem $u(a, x)$, returns the linearized optimum $\tilde{a}$, or a sparse version of it, $\tilde{a} = a^d + \sum_i b_i m_i x_i$

Second, we can say that the agent uses a proxy utility function. We observe that for (a, x) close to $\left(a^d, 0\right)$, we have:

$$
\begin{aligned}
u(a,x) &= u^i(a,x) + O\left(x^2\right) \\
u^1(a,x) &= u^d(a-bx) + u_x\left(a^d, 0\right)x \\
u^2(a,x) &= u^d\left(a^d, x\right) + \frac{1}{2}u^d_{aa}\left(a-a^d\right)^2 + \left(a-a^d\right)u_{ax}x \\
u^3(a,x) &= u^d(a-bx) + u\left(a^d, x\right) - u^d\left(a^d\right)
\end{aligned}
$$

The above representations u^i all "work", i.e. deliver the linear expansion.

15.6.3 Linearizing a relation

Suppose that there's a nonlinearity, say it's $\hat{r}_t = f^r(x)$, e.g. $f^r(x) = \frac{1}{2}x + bx^3$, then the agent may use a linearized policy, i.e.,

$$
f(x,m) = f(0) + m_1 f'(0)x + m_2\left(f(x) - f(0) - f'(0)x\right)
$$

This is, the function is approximated by its constant, first order term (with weight m_1), and higher order terms (with weights m_2). This way, one has a "simpler" representation by linearization.

15.7 Notes on the design of the model

Here I record some notes about modelling choices of the model. This section should only interest people thinking about the foundations of the approach (hence, potential ways to change it), not its direct practical use.

15.7.1 Breaking the explosion of Thinking about thinking about thinking...

Why not model iterated expectations, such as the agent's perception at time 0 of his perception at time 2 of his perception at time 5? The short answer is that this leads to a combinatorial explosion of the complexity of the model. This motivates the particular formulation of sparse dynamic programming, which eschews such a combinatorial explosion.

I record this phenomenon in this subsection, using the simple 3−period model of Section 4.1, extended here to $T+1$ periods. There we obtain about 2^T state variables for a T−period model.

Utility is $\sum_{t=0}^{T} u(c_t)$ and $R = 1$. The agent receives w_0 at time 0, and x at time t. So, $w_t = w_{t-1} - c_{t-1} + x1_{t=T}$ for $t \geq 1$. The rational problem is

$$
\max \sum_{t=0}^{T} u(c_t) \text{ s.t. } \sum_{t=0}^{T} c_t = w_0 + x
$$

and the optimal consumption policy at time t is $c_t = \frac{w_t + x}{T-t+1}$

$$V^T(w_T, x) = u(w_T + x)$$

Next, at time $T-1$, the rational value function is (RE stands for rational expectations):

$$V^{T-1,RE}\left(w_{T-1}, x, M_{T-1}^{RE}\right) = u\left(\frac{w_{T-1} + m_{T-1}^* x}{2}\right) + u\left(\frac{w_{T-1} + \left(2 - m_{T-1}^*\right)x}{2}\right)$$

for the value m_{T-1} chosen at time $T-1$,. However, the perceived value function could be

$$V^{T-1,RE}(w_{T-1}, x, M_{T-1})$$

for some other perceived M_{T-1}.

At time $T-2$, the problem is

$$V^{T-2}\left(w_{T-2}, x, M^{RE}\right) = u(c_{T-2}) + V^{T-1}\left(w_{T-2} - c, x, M_{T-1}^{RE}\right),$$

with

$$c_{T-2} = \arg\max_c u(c) + V^{T-1}\left(w_{T-2} - c, m_{T-2}^x x, E_{T-2}[M_{T-1}]\right)$$

so that at each stage, the agent gets either "M^{RE}" or "M^S". So, the relevant perception vector is $M_{T-2} = \left(m_{T-2}, M_{T-1}, E_{T-2}^s[M_{T-1}]\right) \in \mathbb{R}^3$ – the datum of the perception of x, M_{t-1} and the actual M_{T-1}.

More generally, at time t, the value function is

$$V_t(w_t, x, M_t) = u(c_t^*) + V_{t+1}(w_t - c_t^*, x, M_{t+1})$$

with

$$c_t^* = \arg\max_{c_t} u(c_t) + V_{t+1}(w_t - c_t, m_t^x x, E_t^s[M_{t+1}])$$

So, the perception vector at time t is

$$M_t = (m_t^x, M_{t+1}, E_t^s[M_{t+1}])$$

i.e. it's formed of the attention m_t^x to x, tomorrow's perception vector M_{t+1}, and today's perception about tomorrow's vector, $E_t^s[M_{t+1}]$. Ideally, the agent should keep track of all those. Call $D_t = \dim M_t$. We have $D_T = 0, D_{T-1} = 1$, $D_t = 1 + 2D_{t+1}$. This yields that the dimension of the attention vector at time t is

$$D_t = 2^{T-t} - 1.$$

which is overwhelming.

Simplified value function. This is why the main sparse max cuts through the difficulty to allow for just two value functions: $V^{t,RE}(w_t, x)$, the rational expectation value function, and $V^{t,S}(w_t, x)$, the "simplified value function". Then, we form

$$V^t\left(w_t, x, m_t^V\right) = m_t^V V^{t,RE}(w_t, x) + \left(1 - m_t^V\right) V^{t,S}(w_t, x)$$

Lemma 15.3 *The value function $V^t(w, x, M)$ is independent of M, up to $O(x^2)$ terms.*

Hence, we define the germ $V^t\left(w, x, M^S\right)$ to be a "simplified" version of $V^t(w, x, M)$. In many cases, it's the $V^t\left(w, x, M^S\right) = V^t(w, x, M^r)$, assuming that the agent will be rational. Typical case, $u_c = V_w(w, x, M)$, with $c = w(A + Bx) + C + Dx + o(x)$. (This applies with time-varying interest rate, income and equity premium). Then, we set $V\left(w, x, M^S\right) := u(w(A + Bx) + C + Dx)/(A + Bx)$. If chosen representation is $c = we^{A+Bx} + o(x)$, then we set: $V\left(w, x, M^S\right) := u\left(we^{A+Bx}\right)/e^{A+Bx}$.

More abstractly, if it's $c = \phi(A(w) + B(w)x) + o(x)$, for some function ϕ, then we set $V_w\left(w, x, M^S\right) = u_c(\phi(A(w) + B(w)x))$.

15.8 Generalization: The k, K Procedure

Here I discuss how to do an expansion when V_{xx} is required but not known by the agent. This greatly generalizes the Cass-Koopmans of Section 5.

Suppose that we want to solve

$$V(X_0) = \max_{(a_t)_{t\geq 0}} \sum_{t\geq 0} \beta^t u(X_t, a_t) \text{ s.t. } X_{t+1} = F(X_t, a_t) \tag{136}$$

but do not know $V_{XX}(X_0)$, which is required to find a_{X_0} (recall that the first order condition is $u_a + \beta V_X F_a$, so calculating a_{X_0} involves V_{XX}). What to do?

I posit the following description of the agent's world view and behavior. He considers x_t "his" variable, and X_t the value created by the environment, which is perceived to be exogenous to him: this is the same way that in much of macro, k_t is his wealth, and K_t is the aggregate capital stock. He has a mental model of the law of motion of X_t on the equilibrium path, e.g. as in $X_{t+1} - X_t = B(X_t - X_*)$ for some matrix B, and X_* is the steady state value of X_t. He solves in his mind the "micro" problem:

$$V(x_0, X_0) = \max_{(a_t)_{t\geq 0}} \sum_t \beta^t \bar{u}(x_t, X_t, a_t) \text{ s.t. } x_t = \bar{F}^x(x_t, X_t, a_t) \tag{137}$$

with X_t exogenous to his actions and I define the modified utility and production function, in a manner that separates the micro variable x and the macro variable X :

$$\bar{u}(x, X, a) := u(X, a) + u_X(X, a)(x - X), \qquad \overline{F}^x(x, X, a) := F(X, a) + F_X(X, a)(x - X)$$

This way, they capture the marginal contribution of x.[87,88].

Next, I assume that the agent knows V, V_x, V_{xx} evaluated at $(x, X) = (X_*, X_*)$ – i.e., the agent has solved the "microeconomic" problem of optimizing when the macroeconomic environment is at the steady state, i.e. $X_t = X_*$ at all dates, but his microeconomic variable x_t is off equilibrium.[89] This gives the value $a_x (x, X_*)$. This way, we can calculate $V_{x,X}$ using the procedure in Section 10.1 (using the change in notations $(w, x) := (x, X)$).

Hence the action is $\hat{a} = a_x \hat{x}_t + a_X \hat{X}_t$ and on the equilibrium path $\hat{x}_t = \hat{X}_t$. To calculate a_X we proceed as in section 10.1.[90] Section 12.4 applies this to an optimal real investment problem.

15.9 Continuous time

Calculations are typically cleaner in continuous time, so we develop the continuous-time version of the machinery. We take for now problems without stochastic terms (those should be added later).

The laws of motion are

$$\begin{aligned} \dot{w}_t &= F^w (w, x, a) \\ \dot{x}_t &= F^x (w, x) \end{aligned}$$

and the Bellman equation is

$$\rho V (w, x) = u (w, x, a) + V_w (w, x) F^w (w, x, a) + V_x (w, x) F^x (w, x, a)$$

In the more complex case $\dot{x}_t = F^x (w, x, a)$, we need to solve a matrix Ricatti equation – but

---

[87] One could also imagine using A, the other agent's action, and setting $\overline{F} (x, X, a, A) := F (X, A) + F_X (X, A) (x - X) + F_a (X, A) (a - A)$, or similar variants. What's important is that values and first order derivatives are preserved around $(x, a) = (X, A)$: more precisely, function $\overline{F} (x, X, a, A)$ must satisfy: $\overline{F} (x + \varepsilon, x, a + \eta, a) = F (x + \varepsilon, a + \eta) + O (\varepsilon^2 + \eta^2)$, and similarly $\overline{u} (x, X, a, A)$ must satisfy: $\overline{u} (x + \varepsilon, x, a + \eta, a) = u (x + \varepsilon, a + \eta) + O (\varepsilon^2 + \eta^2)$.

[88] For instance, in the Cass-Koopmans problem, $(x, X, a) := (k, K, c)$, $F (K, c) = f (K) - c$, and

$$\overline{F}^k (k, K, c) = f (K) - c + f' (K) (k - K) = f' (K) k + [f (K) - K f' (K)] - c$$

so that $f' (K) k$ is the return on capital, and $[f (K) - K f' (K)]$ is labor income.

[89] This is actually easy to derive in a number of canonical problems: For instance, in RCK this is saying that the agent knows the "micro" problem of the life-cycle with constant interest rates. This is also true in a canonical real investment problem derived below.

[90] The FOC is $\Psi (x, X, a) = 0$ with

$$\Psi (x, X, a) := \bar{u}_a (x, X, a) + \beta V_x (x, X) \overline{F}^x_a (x, X, a)$$

This gives the marginal impact of X on the action $a_X = -\Psi_a^{-1} \Psi_X$, with

$$\begin{aligned} \Psi_a &= \bar{u}_{aa} + \beta V_x \overline{F}^x_{aa} = u_{aa} + F_{aa} \\ \Psi_X &= \bar{u}_{aX} + \beta V_{xX} \overline{F}^x_a + \beta V_x \overline{F}^x_{aX} = \bar{u}_{aX} + \rho V_{xX} \overline{F}^x_a. \end{aligned}$$

where in the last two equations, the last part of the right-hand side is evaluated at $x = X$.

not here.

Call, for some function f, $D_w f = \partial_w f + a_w \partial_a f$ the "total impact" of a change in w. Then differentiate the Bellman equation with respect to x,

$$\rho V_x = u_x + V_w F_x^w + V_x F_x^x + V_{xx} F^x \tag{138}$$

Now, we differentiate with respect to w and evaluate at $x = 0$:

$$\rho V_{wx} = D_w \left(u_x + V_w F_x^w\right) + V_{wx} F_x^x + V_x F_{wx}^x$$

so

$$V_x = \left(\rho - F_x^x\right)^{-1} \left[u_x + V_w F_x^w\right] \tag{139}$$

$$V_{wx} = \left(\rho - F_x^x\right)^{-1} \left[D_w \left(u_x + V_w F_x^w\right) + V_x F_{wx}^x\right] \tag{140}$$

As a satisfies $\Psi = 0$ with

$$\Psi\left(a, w, x\right) = u_a + V_w F_a^w$$

where we have used here $F_a^x = 0$. Hence, the impact of x on the optimal action is

$$a_x = -\Psi_a^{-1} \Psi_x$$

$$\begin{aligned} \Psi_a &= u_{aa} + V_w F_{aa}^w \\ \Psi_x &= u_{ax} + V_{wx} F_a^w + V_w F_{ax}^w \end{aligned}$$

Calculation of V_{xx}. We now turn to the more difficult case of V_{xx}. Using $D_x f = \partial_x f + a_x \partial_a f$ the "total impact" of a change in x, we have:

$$\begin{aligned} \rho V_x &= D_x u + V_w D_x F^w + V_x F_x^x + V_{xx} F^x \\ &= a_x \left(u_a + V_w F_a^w\right) + u_x + V_w F_x^w + V_x F_x^x + V_{xx} F^x \end{aligned}$$

Next, differentiating at $x = 0$,

$$\begin{aligned} \rho V_{xx} &= a_x D_x \left(u_a + V_w F_a^w\right) + D_x \left[u_x + V_w F_x^w + V_x F_x^x\ \right] + V_{xx} F_x^x \\ &= a_x \left[u_{ax} + u_{aa} a_x + V_{wx} F_a^w + V_w F_{ax}^w + V_w F_{aa}^w a_x\right] \\ &+ u_{xx} + u_{xa} a_x + V_{xw} F_x^w + V_w F_{xx}^w + 2 V_{xx} F_x^x + V_x F_{xx}^x \end{aligned}$$

hence

$$\begin{aligned}(\rho - 2F_x^x)V_{xx} &= a_x\left[u_{ax} + u_{aa}a_x + V_{wx}F_a^w + V_wF_{ax}^w + V_wF_{aa}^w a_x\right] \\ &\quad + u_{xx} + u_{xa}a_x + V_{xw}F_x^w + V_wD_xF_x^w + V_xF_{xx}^x\end{aligned}$$

This is a bit of a complicated expression. Let us note it can be written

$$(\rho - 2F_x^x)\left(V_{xx}^s - V_{xx}^r\right) = a_xA + a_xBa_x + C$$

with $B = u_{aa} + V_wF_{aa}^w$.

We use the following elementary Lemma:

Lemma 15.4 *Let $f(a) = Aa + a'Ba + C$, for B symmetric negative definite. Let $a^* = \arg\max_a f(a)$, so $a^* = -\frac{1}{2}B^{-1}A$. Then, for any a,*

$$f(a) - f(a^*) = (a - a^*)\,B\,(a - a^*)\,.$$

Let's compare V_{xx} under the sparse vs rational model: the difference is just in the D_x^r vs D_x^s term. Indeed,

$$D_x^s - D_x^r = \left(a_x^s - a_x^r\right)\partial_a$$

so, using the previous Lemma,

$$V_{xx}^s - V_{xx}^r = (\rho - 2F_x^x)^{-1}\left(a_x^s - a_x^r\right)\left(u_{aa} + V_wF_{aa}^w\right)\left(a_x^s - a_x^r\right) \tag{141}$$

We gather the results.

Proposition 15.5 (What are the losses from a suboptimal policy?) *Consider the value function V^r under the optimal policy and V^s under a potentially suboptimal policy, and $V^\delta(w,x) = V^s(w,x) - V^r(w,x)$. Then, evaluating at $x = 0$, we have*

$$V^\delta = 0, V_w^\delta = 0, V_{ww}^\delta = 0, V_x^\delta = 0, V_{wx}^\delta = 0 \tag{142}$$

and

$$V_{xx}^\delta = (\rho - 2F_x^x)^{-1}\left(a_x^s - a_x^r\right)\left(u_{aa} + V_wF_{aa}^w\right)\left(a_x^s - a_x^r\right) \tag{143}$$

Equation (143) has an intuitive interpretation. At a point in time, as a function of a, present and continuation utility is $v(a) = u(a, w_t)\,dt + (1 - \rho dt)\,V\left(w_t + F^w(w_t, a_t)\,dt\right)$. Hence (omitting the dt to remove the notational clutter), $v'(a) = u_a + V_wF_a^w$ and $v''(a) = u_{aa} + V_wF_{aa}^w$. Hence, reacting imperfectly to a small x_t (with $a_t^\delta = a_t^s - a_t^r$) creates an instantaneous utility loss of

$\Lambda_t = -\frac{1}{2}x_t a_x^\delta v_{aa} a_x^\delta x_t$. The full utility loss is the present discounted value of that, i.e.

$$
\begin{aligned}
2\Lambda &= \int_0^\infty e^{-\rho t} 2\Lambda_t dt = -\int_0^\infty e^{-\rho t} x_t a_x^\delta v_{aa} a_x^\delta x_t \text{ with } x_t = e^{-\phi t} x_0 \\
&= -\int_0^\infty e^{-\rho t} e^{-2\phi t} x_0 a_x^\delta v_{aa} a_x^\delta x_0 = \frac{1}{\rho + 2\phi} x_0 a_x^\delta v_{aa} a_x^\delta x_0 \\
&= -x_0 \left(\rho - 2F_x^x\right)^{-1} a_x^\delta \left(u_{aa} + V_w F_{aa}^w\right) a_x^\delta x_0 \text{ as } F_x^x = -\phi \\
&= -x_0 V_{xx}^\delta x_0.
\end{aligned}
$$

It is enough to study the "static" utility losses to derive the dynamic utility losses. This proposition 15.5 is a dynamic application of the Proposition 26 in Gabaix (2014, online appendix) regarding losses from a suboptimal policy. For convenience, we restate this proposition here. With static problem $\max u(a,x)$ s.t. $b(a,x) \geq 0$ and a Lagrangian $L(a,x) = u(a,x) + \lambda b(a,x)$, the losses from a suboptimal policy $a^\delta = a - a^r$ (where a^r is the optimal policy) are to the leading order: $\frac{1}{2} a^{\delta\prime} L_{aa} a^\delta$.

Here the Lagrangian is $L = \int e^{-\rho t}\left[u(a_t, z_t) + \lambda_t\left(-\dot{z}_t + F^z(a_t, z_t)\right)\right] dt$, where $z_t = (w_t, x_t)$ is the state vector. Hence, the loss Λ is expressed by (to the leading order)

$$
2\Lambda = a' L_{aa} a = \int a_t^\delta L_{a_t a_t} a_t^\delta = \int e^{-\rho t} a_t^\delta \left[u_{a_t a_t} + \lambda_t F_{a_t a_t}\right] a_t^\delta dt
$$

Suppose that we can linearize, $a_t^\delta = A x_t$, we have

$$
2\Lambda = \int e^{-\rho t} x_t' A' \left[u_{a_t a_t} + \lambda_t F_{a_t a_t}\right] A x_t dt
$$

Consider the ergodic limit, where x_t has a distribution independent of t. Recall that

$$
\mathbb{E} x_t' B x_t = \mathbb{E} \sum_{i,j} x_i B_{ij} x_j = \sum_{i,j} B_{ij} \mathbb{E}\left[x_i x_j\right] = \text{Trace}\left(B \mathbb{E}\left[x x'\right]\right)
$$

Hence,

$$
\begin{aligned}
2\Lambda &= \frac{1}{\rho} \text{Trace}\left(B \mathbb{E}\left[x x'\right]\right) \\
B &= A' \left[u_{a_t a_t} + \lambda_t F_{a_t a_t}\right] A = A' L_{a_t a_t} A
\end{aligned}
$$

15.10 Some ancillary results

Call $G(Z, m) = Z(m)$ a transformation function for the state vector Z. E.g. in the basic life-cycle example, $G(w, \hat{y}, \hat{r}, m) = (w, m_y \hat{y}, m_r \hat{r})$. [Note: below, the notation bar isn't ideal, as bar refers to means; perhaps tilde would be better] When can we express the perceived model as a rational model, with different utility and transition functions? The following Lemma gives the answer.

Lemma 15.6 *Let $G(Z,m)$ be a function and define $\bar{Z}_t = G(Z_t, m)$. Suppose that we can write*

$$u(a,Z,m) = \bar{u}(z, G(Z,m))$$
$$G(F(a,Z,m),m) = \bar{F}(a, G(Z,m))$$

for two functions $\bar{u}, \bar{F}$. Then the model evaluated at m is the same as a rational model with state variables $\bar{Z}_t$, utility $\bar{u}$, transition function $\bar{F}$. We also have $V(Z) = \bar{V}(G(Z,m))$

Proof We have

$$\bar{Z}_{t+1} = G(Z_{t+1}, m) = G(F(a, Z_t, m), m) = \bar{F}(a, G(Z_t, m)) = \bar{F}\left(a, \bar{Z}_t\right)$$

The value function $\bar{V}$ satisfies, with Z s.t. $\bar{Z} = G(Z,m)$

$$\begin{aligned}\bar{V}\left(\bar{Z}\right) &= \max_a \bar{u}\left(a, \bar{Z}\right) + \beta\bar{V}\left(\bar{F}\left(a, \bar{Z}\right)\right) \\ &= \max u(a,Z,m) + \beta\bar{V}(G(F(a,Z,m),m))\end{aligned}$$

Define $V(Z) := \bar{V}(G(Z,m))$. Then,

$$V(Z) = \bar{V}\left(\bar{Z}\right) = \max_a u(a,Z,m) + \beta V(F(a,Z,m),m)$$

So indeed V satisfies the Bellman equation. $\square$

Also, we have $F^w(Z,m) = ((R+\bar{r})(w-\bar{y}), \rho)$

$$\hat{y}_{t+1} = m_F \rho_y \hat{y}_t + \varepsilon^y_{t+1}$$

gives

$$m_y \hat{y}_{t+1} = m_{y'}\rho_y\left(m_y\hat{y}_t + m_y\varepsilon^y_{t+1}\right)$$

i.e.

$$\bar{F}^{\bar{y}}\left(a, \bar{Z}\right) = m_{y'}\rho_y\left(\bar{y}_t + m_y\varepsilon^y_{t+1}\right)$$

I conclude with a remark which will be useful later, drawing again on Gabaix (2014). As κ has the units of utils, one can make it more endogenous with the primitive, unitless parameter $\overline{\kappa}$, by setting:

$$\kappa = \overline{\kappa}^2 var\left(u\left(a^d(x), x, m^d\right)\right)^{1/2} \tag{144}$$

16 Other Results

16.1 Second Order Losses From Inattention

16.1.1 Statement

The intuition is that as x is small, and the action is close to the optimum, we get only second order losses $O\left(\|x\|^2\right)$ from misoptimization. The idea is simple, but it turns out that it requires some extra care, in particular to ensure that differentiability, and to define formally the objects of interest. I present that here. This subsection does not contain surprising results, so should be skipped at the first reading.

Recall that w is a set of variables thought about in the default model ($m = 0$), and x is a set of variable not thought about in the default model. We also set $z = (w, x)$. Formally, we assume that $u^t(a, w, x, m)$ and $F^t(a, w, x, m)$ are independent of m if $x = 0$. We also assume that $F^{x,t}(a, w, x, m) = 0$ if $x = 0$, i.e. so that small x's at t map in small x's at $t+1$.

We suppose that the attention function $\mathcal{A}$ is C^∞ (this can easily be ensured).

Recapitulations of the notations *With 1 period problems*, the action is

$$a(z, m) := \arg\max_a u(a, z, m) \tag{145}$$

With exogenous attention m, the value function is the utility evaluated at the true model of the world:

$$V(z, m, \mu) := u(a(z, m), z, \mu) \tag{146}$$

where μ denotes the true state of the world (typically $\mu = (1, ..., 1)$). With endogenous attention, we have

$$m_i^*\left(w, \sigma_x^2, \mu\right) := \arg\max_{m_i \in [0, \mu_i]} E\left[\frac{1}{2} V_{m_i m_i}(w, x, m)_{|m=0} (m_i - \mu_i)^2 + G(m_i)\right] \tag{147}$$

and the value function with endogenous attention is

$$V(z, \mu) := V\left(z, m^*\left(w, \sigma_x^2, \mu\right), \mu\right) \tag{148}$$

We sometimes drop the explicit dependence on μ.

With T period problems, the notions are the same, but recursive. We use $m = (m_t)_{t=0...T}$. The action at time t is:

$$a^t(z, m) := \arg\max_a u^t(a, z, m_t) + \beta V^{t+1}\left(F^t(a, z, m_t), m_{t+1}\right) \tag{149}$$

which defines the value function, evaluated at the true model of the world. With exogenous attention:

$$V^t(z,m,\mu) := u^t\left(a^t(z,m), z, \mu_t\right) + \beta V^{t+1}\left(F^t(a,z,m_t), \mu_{t+1}\right) \tag{150}$$

When attention is endogenous, we have

$$m_i^{t,*}\left(w,\sigma_x^2,\mu\right) := \arg\max_{m_{t,i}\in[0,\mu_{t,i}]} E\left[\frac{1}{2}V^t_{m_i m_i}(w,x,m)_{|m_t=0}\left(m_{t,i}-\mu_{t,i}\right)^2 + G\left(m_{t,i}\right)\right] \tag{151}$$

and

$$V^t(z,\mu) := V^t\left(z, m^{t,*}\left(w,\sigma_x^2,\mu\right),\mu\right). \tag{152}$$

We call $V^r(w,x) := V^t\left(w,x,(\mu_t,\mu_t)_{t=0...T}\right)$ the rational value function.

We prove two propositions: the first one is elementary to state, the other one a bit more cumbersome.

Proposition 16.1 (Second order losses form inattention) *Suppose that u^t and F^t are C^∞. Recall the decomposition $z=(w,x)$. Then, with exogenous m, $a^t\left(z,(m_\tau,\mu_\tau)_{\tau=t...T}\right)$ and $V^t\left(z,(m_\tau,\mu_\tau)_{\tau=t...T}\right)$ are C^∞ and,*

$$V^t\left(w,x,(m_\tau,\mu_\tau)_{\tau=t...T}\right) - V^{t,r}(w,x) = O\left(\|x\|^2\right)O\left(\sum_{\tau=t}^T\|m_\tau-\mu_\tau\|^2\right) \tag{153}$$

With endogenous m, $a^t(z)$ and $V^t(z)$ are C^∞, and

$$V^t(w,x) - V^{t,r}(w,x) = O\left(\|x\|^2\right) \tag{154}$$

Proposition 16.2 (Second order losses form inattention, with finite differentiability*) With exogenous m, assume that there is an $\ell \geq 3$ such that u^t is $C^{t+\ell}$, F^t (the transition function from t to $t+1$) is $C^{t+\ell}$. Then $a^t\left(z,(m_\tau,\mu_\tau)_{\tau=t...T}\right)$ and $V^t\left(z,(m_\tau,\mu_\tau)_{\tau=t...T}\right)$ are $C^{t+\ell-1}$ for all $t=0,...,T$, and (153) holds.*

With endogenous m, assume that there is an $\ell \geq 5$ such that u^t $(t\leq T)$, F^t (the transition from t and $t+1$) are $C^{3t+\ell}$ for some ℓ. Then $a^t(z)$ and $V^t(z)$ are $C^{3t+\ell-3}$ and (154) holds.

16.1.2 Proof of Propositions 16.1 and 16.2

The proof starts with the most elementary cases (1 period), then amplifies it to 2 and more periods.

1 period, with exogenous m We suppose that $u(a,w,x,m)$ is C^k, with $k\geq 4$, and that the rational problem is

$$\max_a u(a,w,x,m)$$

We shall see that $V(w,x,m)$ is C^{k-1} and $V(w,x)$ is C^{k-3}.

First, let us suppress the dependence on w, and consider $\max_a u(a, x, m)$.

By the implicit function theorem, $a(x, m) := \arg\max_a u(a, x, m)$ is C^{k-1} (it is the solution of $u_a(a, x, m) = 0$). This makes $V(x, m, \mu) := u(a(x, m), x, \mu)$ be C^{k-1}.

As $u(a, 0, m)$ is independent of m, $a(0, m)$ is independent of m, so we can write: $a(x, m) = a(0, 0) + b(m)x + O(x^2)$. As $a_x(x, m)$ is C^{k-2} with $k - 2 \geq 1$ there is a constant L such for that all m, μ in the compact $[0, 1]^n$, we have $\|b(m) - b(\mu)\| \leq L\|m - \mu\|$. We consider the loss from imperfect perception:

$$\begin{aligned}
R(x, m, \mu) &:= V(x, m, \mu) - V(x, \mu, \mu) \\
&= u(a(x, m), x, \mu) - u(a(x, \mu), x, \mu) \\
&= (a(x, m) - a(x, \mu))' u_{aa}(a(x, m) - a(x, \mu)) + o(a(x, m) - a(x, \mu))^2 \\
&= (b(m) - b(\mu))' x u_{aa} x (b(m) - b(\mu))' + o\left(\|x\|^2\right)
\end{aligned}$$

So we have

$$V(x, m, \mu) - V(x, \mu, \mu) = R(x, m, \mu) = O\left(\|x\|^2\right) O\left(\|m - \mu\|^2\right) \tag{155}$$

Reinserting the dependence in w, the same reasoning show that with

$$R(w, x, m, \mu) := V(w, x, m, \mu) - V(w, x, \mu, \mu) \tag{156}$$

we have

$$R(w, x, m, \mu) = O\left(\|x\|^2\right) O\left(\|m - \mu\|^2\right). \tag{157}$$

1 period, with endogenous m We supposed that $u(a, w, x, m)$ was C^k, which led $V(w, x, m)$ to be C^{k-1}. Now, we endogenize m. We have

$$m_i^*\left(w, \sigma_x^2\right) = \arg\max_m E\left[-\frac{1}{2} V_{m_i m_i}(w, x, m)_{|m=0} (m_i - \mu_i)^2 - G(m_i)\right]$$

So, $m^*(w)$ is a C^{k-3} function of w. So $V(w, x) := V(w, x, m^*(w))$ is a C^{k-3} function.

Hence, starting from a C^k function $u(a, w, x, m)$, we obtain a C^{k-3} value function $V(w, x)$. We "lost" 3 orders of differentiability.[91]

2 period problems There are 2 periods, 0 and 1. We call

$$\begin{aligned}
V^1(w, x, m, \mu) &:= u^1(a(w, x, m), w, x, \mu) \\
V^1(w, x, \mu) &:= u^1\left(a\left(w, x, m^*\left(w, \sigma_x^2\right)\right), w, x, \mu\right)
\end{aligned}$$

[91] If there was no w, then we'd just have m^* a value independent of x, and function $V(w, x)$ would be C^{k-1}.

the value function at the beginning of period 1, with respectively exogenous and endogenous attention.

We assume that u^1 is C^k.

2 periods, with exogenous m_1 The problem is now

$$\max_a v^0 (a, w, x, m_0, m_1, \mu_1)$$

with

$$v^0 (a, x, m_0, m_1, \mu_1) := u^0 (a, x, m_0) + V^1 (F (a, w, x, m_0), m_1, \mu_1)$$

where the last function $V^1 (w, x, m_1)$, which is C^{k-1}, and $F = (F^w, F^x)$ gives the transition functions for both w and x. We assume that F, V^1, u^0 are C^{k-1}. So, function v^0 is C^{k-1}.

The reasoning in the 1 period case applies, and $a^0 (w, x, m_0, m_1, \mu_1)$ is C^{k-2}, so

$$V^0 (x, m_0, \mu_0, m_1, \mu_1) := v^0 \left(a^0 (x, m_0, m_1, \mu_1), x, \mu_0, m_1, \mu_1\right)$$

is C^{k-2}. If $k \geq 3$, we have

$$V^0 (x, m_0, \mu_0, m_1, \mu_1) - V^0 (x, \mu_0, \mu_0, m_1, \mu_1) = O\left(\|x\|^2\right) O\left(\|m_0 - \mu_0\|^2\right)$$

by the time-0 result.

Also, we have

$$V^0 (x, \mu_0, \mu_0, m_1, \mu_1) - V^0 (x, \mu_0, \mu_0, \mu_1, \mu_1) = O\left(\|x\|^2\right) O\left(\|m_1 - \mu_1\|^2\right)$$

so putting summing the two differences:

$$V^0 (x, m_0, \mu_0, m_1, \mu_1) - V^0 (x, \mu_0, \mu_0, \mu_1, \mu_1) = O\left(\|x\|^2\right) O\left(\|m_0 - \mu_0\|^2 + \|m_1 - \mu_1\|^2\right)$$

2 periods, with endogenous m_0 The problem is now

$$\max_a v^0 (a, w, x, m_0)$$

with

$$v^0 (a, w, x, m_0) := u^0 (a, w, x, m_0) + V^1 (F (a, w, x, m_0))$$

where the last function is $V^1 (w, x)$, which is C^{k-3}, and $F = (F^w, F^x)$ gives the transition functions for both w and x. We assume that F, V^1, u^0 are C^{k-3}. So, function $v^0 (a, x, m_0)$ is C^{k-3}.

By the reasoning before with 1 period, $a(w,x,m_0)$ is C^{k-4} and

$$V^0(a,w,x,m_0) = v^0(a(w,x,m_0),w,x,m_0)$$

is C^{k-4}. Next, to endogenize m_0, again by the reasoning done with 1 period, $m_0(w,\sigma_x^2)$ is C^{k-6}, so that

$$V^0(a,w,x) = V^0\left(a,w,x,m_0^*\left(w,\sigma_x^2\right)\right) \tag{158}$$

is C^{k-6}.

We next move to more than 2 periods. The reasoning is very similar to the 2 period case.

$T+1$ **periods, exogenous** m We assumed that u^t is $C^{t+\ell}$, F^t (from t to $t+1$) is $C^{t+\ell}$ for some $\ell \geq 3$. That implies that a^T is $C^{T+\ell-1}$, V^T is $C^{T+\ell-1}$, and by backward induction on $t = T...0$, that a^t and V^t are $C^{t+\ell-1}$ for all $t = 0,...,T$. So, if $\ell \geq 3$, then V^0 is C^2, and

$$V^0\left(w,x,(m_t,\mu_t)_{t=0...T}\right) - V^0\left(w,x,(\mu_t,m_t)_{t=0...T}\right) = O\left(\|x\|^2\right)O\left(\sum_{t=0}^{T}\|m_t-\mu_t\|^2\right) \tag{159}$$

holds as well.

$T+1$ **periods, endogenous** m We assumed that u^t $(t \leq T)$, F^t (the transition from t and $t+1$) is $C^{3t+\ell}$ for some ℓ. Then, by backwards induction on $t = T...0$, V^t is $C^{3t+\ell-3}$. Indeed, by the reasoning done in the 2 period case, given that u^t is C^k, we have that V^t is C^{k-3}. As F^{t-1}, u^{t-1} are assumed to be C^{k-3}, we have that V^{t-1} is C^{k-6}.

This ends the proof of Propositions 16.1 and 16.2. □

16.1.3 Extensions

Noise In the problems above, there is no noise. Adding noise is straightforward, but adds yet another layer of notations. Formally, we assume bounded noise:

$$\left\|\varepsilon_{t+1}^x\right\| \leq K\sigma_x \text{ almost surely}$$

for some K, and where $\sigma_x = E\left[\|x\|^2\right]^{1/2}$. Then, the statements in Propositions 16.1 and 16.2 are replaced by

$$V^t\left(w,x,(m_\tau,\mu_\tau)_{\tau=t...T}\right) - V^{t,r}(w,x) = O\left(\|x\|^2+\sigma_x^2\right)O\left(\sum_{\tau=t}^{T}\|m_\tau-\mu_\tau\|^2\right) \tag{160}$$

and

$$V^t(w,x) - V^{t,r}(w,x) = O\left(\|x\|^2+\sigma_x^2\right) \tag{161}$$

This is, both the actual value and the variance of x matter.

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