WEB APPENDIX

This appendix summarizes the microfoundations of the simple general equilibrium model of section 4.

A Households

Individuals live for two periods, young and old, and maximize utility from consumption of one aggregate good according to:

$$U_t(c_t^y, c_{t+1}^o) = \max_{c_t^y, c_{t+1}^o} \mathbb{E}_t \left\{ u(c_t^y) + \beta u(c_{t+1}^o) \right\} \tag{1}$$

$$\text{s.t. } c_t^y = w_t l_t - \tau_t - s_t \tag{2}$$

$$c_{t+1}^o = \frac{(1+i_t)}{\Pi_{t+1}} s_t \tag{3}$$

where the $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ is a constant relative risk aversion (CRRA) preference function. c_t^y and c_{t+1}^o are household's consumption respectively when young and old. When young, individuals earn income in period t by renting their labor endowment l_t to firms at wage w_t. After paying taxes τ_t the young use their net income to consume in period t and to save s_t for consumption when old by accumulation of private capital supplied to firms for production during the next period for a gross real rent $\frac{(1+i_t)}{\Pi_{t+1}}$, such that:

$$K_{t+1}^s = N_t^y s_t \tag{4}$$

where N_t^y is the size of young generation at time t. When old, individuals dissave to consume, earning a gross real return $\frac{(1+i_t)}{\Pi_{t+1}}$ on their savings from previous period (3). We derive the first order conditions of this problem by maximizing the Lagrangian[1]:

$$\mathcal{L}_t = u(c_t^y) + \beta u(c_{t+1}^o) - \lambda_t \left(c_t^y - w_t l_t + \tau_t + s_t\right) - \lambda_{t+1} \left(c_{t+1}^o - \frac{(1+i_t)}{\Pi_{t+1}} s_t \right) \tag{5}$$

First-order conditions:

$$\frac{\delta \mathcal{L}_t}{\delta c_t^y} = u_c(c_t^y) - \lambda_t = 0 \tag{6}$$

$$\frac{\delta \mathcal{L}_t}{\delta c_{t+1}^o} = \beta u_c(c_t^o) - \lambda_{t+1} = 0 \tag{7}$$

$$\frac{\delta \mathcal{L}_t}{\delta k_{t+1}^s} = -\lambda_t + \lambda_{t+1} \frac{(1+i_t)}{\Pi_{t+1}} = 0 \tag{8}$$

Perfect foresight young individuals are at an interior solution and their consumption-saving choices satisfy a standard Euler equation given by

$$\lambda_t = \lambda_{t+1} \frac{(1+i_t)}{\Pi_{t+1}} \rightarrow u_c(c_t^y) = \beta R_t u_c(c_t^o) \Leftrightarrow \frac{1}{(c_t^y)^\sigma} = \beta \frac{(1+i_t)}{\Pi_{t+1}} \frac{1}{\left(c_{t+1}^o\right)^\sigma} \tag{9}$$

[1] The expectations operator is ignored since the model is deterministic.

Let $R_t \equiv \frac{(1+i_t)}{\Pi_{t+1}} \equiv 1 + r_t$. Then the previous expression can be written as

$$\frac{1}{c_t^y} = \beta_{R_t,\sigma} R_t \frac{1}{c_{t+1}^o} \Leftrightarrow c_{t+1}^o = R_t \left[\beta_{R_t,\sigma} c_t^y\right] \tag{10}$$

where $\beta_{R_t,\sigma} = \beta^{\frac{1}{\sigma}} R_t^{\frac{1-\sigma}{\sigma}} \overset{(\sigma=1)}{=} \beta$. Directly from the budget constraint of the old(3) we have

$$s_t = \beta_{R_t,\sigma} c_t^y \tag{11}$$

Savings of the young s_t can then be derived by replacing the previous expression of c_t^y with respect to s_t in the budget constraint of the young(2):

$$s_t = \frac{\beta_{R_t,\sigma}}{1+\beta_{R_t,\sigma}} (w_t l_t - \tau_t) \tag{12}$$

Capital supply:

Because aggregate savings in period t is equal to the capital supplied in the following period, we have:

$$N_t^y s_t = K_{t+1}^s \Leftrightarrow s_t = \frac{K_{t+1}^s}{N_t^y} = \frac{K_{t+1}^s}{N_{t+1}^y} \frac{N_{t+1}^y}{N_t^y} = k_{t+1}^s (1+g_t) = \frac{k_{t+1}^s}{A_t} \Rightarrow k_{t+1}^s = A_t s_t \tag{13}$$

where k_t^s is capital supplied per young individual at time t, $1 + g_t = N_{t+1}^y / N_t^y$ is population growth rate, and defining an aging parameter as the ratio of old to young at time $t+1$:

$$A_t = \frac{N_{t+1}^o}{N_{t+1}^y} = \frac{N_t^y}{N_{t+1}^y} = \frac{1}{1+g_t} \tag{14}$$

Then,

$$k_{t+1}^s = A_t s_t = A_t \frac{\beta_{R_t,\sigma}}{1+\beta_{R_t,\sigma}} (w_t l_t - \tau_t) \tag{15}$$

No-arbitrage condition:

The return on savings R_t accounts for the rent R_{t+1}^k on capital firms pay to individuals, and a capital depreciation δ. So, the budget constraint of the old can alternatively be expressed by:

$$c_{t+1}^o = \frac{1}{A_{t+1}} \left[(1-\delta) k_{t+1}^s + R_{t+1}^k k_{t+1}^s\right] = s_t (1 - \delta + r_{t+1}^k) \tag{16}$$

Implying the following no-arbitrage condition:

$$R_{t+1}^k = R_t + \delta - 1 \tag{17}$$

B Firms

We assume that firms produce only one good, are perfectly competitive, and take prices as given. They hire labor at a wage w_t and rent capital at rate r_t^k to maximize period-by-period profits. They operate using a standard Cobb-Douglas production function, and their problem is given by:

$$\max_{L_t,K_t} P_t Y_t - W_t L_t - P_t R_t^k K_t \tag{18}$$

$$\text{s.t. } Y_t = L_t^{1-\alpha} K_t^{\alpha} \tag{19}$$

The firm's capital and labor demand equilibrium conditions are given by:

$$R_t^k = \alpha \frac{Y_t}{K_t} \tag{20}$$

$$w_t = \frac{W_t}{P_t} = (1-\alpha)\frac{Y_t}{L_t} \tag{21}$$

Each individual of the young generation supplies his labor endowment inelastically at $\bar{l}$. Since for now we are assuming wages are flexible, and full-employment, then $L_t = N_t^y \bar{l}$. Let $k_t^d = \frac{K_t}{N_t^y} = \frac{K_t}{L_t}\bar{l}$. Then:

$$w_t = (1-\alpha)\left(\frac{\alpha}{R_t^k}\right)^{\frac{\alpha}{1-\alpha}} \tag{22}$$

$$k_t^d = \bar{l}\left(\frac{\alpha}{R_t^k}\right)^{\frac{1}{1-\alpha}} \tag{23}$$

Defining $\tilde{x} \equiv \ln x$:

$$\tilde{k}_{t+1}^d = \ln\left[\bar{l}\alpha^{\frac{1}{1-\alpha}}\right] - \frac{1}{1-\alpha}\tilde{R}_{t+1}^k \tag{24}$$

C Government

We assume the Government budget is balanced, $G_t = T_t$. And that Government spending is exogenously proportional to full-employment output $G_t = \Omega \bar{Y}_t$.

$$G_t = \mathcal{G}\bar{Y}_t = T_t = N_t^y \tau_t \tag{25}$$

$$\tau_t = \frac{\mathcal{G}}{N_t^y}\bar{Y}_t = \frac{\mathcal{G}}{N_t^y}\frac{w_t \bar{L}_t}{1-\alpha} = w_t \bar{l}\frac{\mathcal{G}}{1-\alpha} = w_t \bar{l}\tau \tag{26}$$

$$\text{where } \tau = \frac{\mathcal{G}}{1-\alpha} \text{ is exogenously determined.} \tag{27}$$

Capital supply per young individual can then be expressed by:

$$k_{t+1}^s = A_t \frac{\beta_{R_t,\sigma}}{1+\beta_{R_t,\sigma}} w_t \bar{l}(\mu_t - \tau) \text{ , where } \mu_t = l_t/\bar{l} \overset{(l_t=\bar{l}_t)}{=} 1 \tag{28}$$

μ_t is the employment ratio of the young, equal to 1 for now. Replacing w_t by (22) and taking logs the previous expression becomes:

$$\tilde{k}^s_{t+1} = \ln\left[\bar{l}(1-\tau)(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\right] + \ln\left(\frac{\beta_{R_t,\sigma}}{1+\beta_{R_t,\sigma}}\right) - \frac{\alpha}{1-\alpha}\tilde{R}^k_t + \tilde{A}_t \tag{29}$$

D Comparative statics

Without loss of generality we assume full depreciation of capital in one period $\delta = 1 \Rightarrow R_t = R^k_{t+1}$. Assuming the system is on a steady state equilibrium where $R_t = R$,

$$\tilde{k}^d = \tilde{k}^s \tag{30}$$

where,from (24) and (29)

$$\tilde{k}^d = -\frac{1}{1-\alpha}\tilde{R} + \ln\left[\bar{l}\alpha^{\frac{1}{1-\alpha}}\right] \tag{31}$$

$$\tilde{k}^s = -\frac{\alpha}{1-\alpha}\tilde{R} + \tilde{A} + \ln\left(\frac{\beta_{R,\sigma}}{1+\beta_{R,\sigma}}\right) + \ln\left[\bar{l}(1-\tau)(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\right] \tag{32}$$

(i) If $\sigma = 1$ then $\beta_{R,\sigma} = \beta$ and $\tilde{R}$ and $\tilde{k}$has the following closed form expression

$$\tilde{R} = -\tilde{A} + \ln\left[\left(\frac{1+\beta}{\beta}\right)\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{1-\tau}\right)\right] \tag{33}$$

$$\tilde{k} = \frac{1}{1-\alpha}\tilde{A} + \frac{1}{1-\alpha}\ln\left[\left(\frac{1+\beta}{\beta}\right)\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{1-\tau}\right)\right] + \ln\left[\bar{l}\alpha^{\frac{1}{1-\alpha}}\right] \tag{34}$$

(ln)Aging $\tilde{A}$ has a one for one negative impact on $\tilde{R}$

$$\frac{d\tilde{R}}{d\tilde{A}} = -1 \tag{35}$$

(ii) For the general case where $\sigma > 0$ we can use the Theorem of the Implicit Function to express the former derivative

$$\frac{d\tilde{R}}{d\tilde{A}} = -\frac{1+\beta_{R,\sigma}}{\frac{1}{\sigma}+\beta_{R,\sigma}} < 0 \tag{36}$$

which is still negative (and equal to -1 when $\sigma = 1$). Also, aging has a stronger impact on real rates when the Relative Risk Aversion σ is higher. Aging expands the supply of capital which effect has to be offset by a reduction of the real rate in order to sustain a general equilibrium. This real rate change has to be higher if the Elasticity of Intertemporal Substution is lower (or σ higher). This is consistent with the data used.

(iii) Impact of aging on output per capita $\tilde{y}^{pc}$

Let,

$$y_t = \frac{Y_t}{L_t} = \left(\frac{K_t}{L_t}\right)^\alpha \Rightarrow \tilde{y}_t = \alpha\tilde{k}_t \tag{37}$$

Since we are assuming full-employment $L_t = N_t^y$. Then,

$$y_t^{pc} = \frac{Y_t}{N_t^y + N_t^o} = \frac{Y_t}{N_t^y} \frac{N_t^y}{N_t^y + N_t^o} = \frac{Y_t}{L_t} \frac{1}{1 + \frac{N_t^o}{N_t^y}} = y_t \frac{1}{1 + A_t} \tag{38}$$

using logs,

$$\tilde{y}_t^{pc} = \tilde{y}_t - \ln(1 + A_t) \tag{39}$$

replacing $\tilde{y}_t = \alpha \tilde{k}_t$

$$\tilde{y}_t^{pc} = \alpha \tilde{k}_t - \ln(1 + A_t) \tag{40}$$

now replacing $\tilde{k}^d = \ln\left[\bar{l}\alpha^{\frac{1}{1-\alpha}}\right] - \frac{1}{1-\alpha}\tilde{R}$

$$\tilde{y}_t^{pc} = -\frac{\alpha}{1-\alpha}\tilde{R}_t - \ln(1 + A_t) + \alpha \ln\left[\bar{l}\alpha^{\frac{1}{1-\alpha}}\right] \tag{41}$$

Finally by replacing $\tilde{R}$ by its steady state expression and taking the derivative of $\tilde{y}_t^{pc}$ with respect to $\tilde{A}$

$$\frac{d\tilde{y^{pc}}}{d\tilde{A}} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1 + \beta_{R,\sigma}}{\frac{1}{\sigma} + \beta_{R,\sigma}}\right) - \left(\frac{A}{1+A}\right) \tag{42}$$

The first term of the expression is the capital deepening effect of aging which is positive, and the second one is the negative demographic effect of aging. Aging has a positive impact on output per capita when the capital deepening effect prevail over the demographic effect:

$$\frac{d\tilde{y^{pc}}}{d\tilde{A}} > 0 \Leftrightarrow \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1 + \beta_{R,\sigma}}{\frac{1}{\sigma} + \beta_{R,\sigma}}\right) > \left(\frac{A}{1+A}\right) \tag{43}$$

We see directly from this expression that for greater values of σ the capital deepening effect is stronger, such that we would expect a stronger positive impact of aging on output per capita in those countries. Note also that the demographic effect $\frac{A}{1+A} = \frac{N^o}{N^y + N^o}$, so in countries where people live longer we would expect a weaker positive relation between aging and output per capita. This is suggested by the data where the significance of the results for OECD countries is much weaker.

E Transition dynamics

Define

$$\tilde{x}^* \equiv \textit{ steady state of } \ln(x) \tag{44}$$

$$\hat{x}_t \equiv \tilde{x} - \tilde{x}^* \tag{45}$$

then from (24) and (29), and having $R_t = R_{t+1}^k$,

$$\hat{k}_{t+1}^d = -\frac{1}{1-\alpha}\hat{R}_{t+1}^k \tag{46}$$

$$\hat{k}_{t+1}^s = -\frac{\alpha}{1-\alpha}\hat{R}_t^k + \hat{A}_t + \left[\ln\left(\frac{\beta_{R_{t+1}^k,\sigma}}{1 + \beta_{R_{t+1}^k,\sigma}}\right) - \ln\left(\frac{\beta_{R^*,\sigma}}{1 + \beta_{R^*,\sigma}}\right)\right] \tag{47}$$

Equilibrium

$$\hat{k}_t^d = \hat{k}_t^s \tag{48}$$

$$\hat{R}_{t+1}^k = \alpha \hat{R}_t^k - (1-\alpha)\hat{A}_t - (1-\alpha)\left[\ln\left(\frac{\beta_{R_{t+1}^k,\sigma}}{1+\beta_{R_{t+1}^k,\sigma}}\right) - \ln\left(\frac{\beta_{R^*,\sigma}}{1+\beta_{R^*,\sigma}}\right)\right] \tag{49}$$

Transition from one steady state to another. Initial steady state: at $t = t_o - 1$ aging $A_{t_o-1} = A_1^*$ and $R_{t_0-1} = R_1^* = R_{t_0}$. At $t = t_0$ aging changes for a change in g from A_1^* to A_2^*. Define $\hat{A}^* \equiv \tilde{A}_1^* - \tilde{A}_2^*$, $\hat{R}^{k*} \equiv \tilde{R}_1^{k*} - \tilde{R}_2^{k*}$, and $\hat{R}_t^k \equiv \tilde{R}_t^k - \tilde{R}_2^{k*}$.

(i) $\sigma = 1$ and $\delta = 1$:

$$\hat{R}_{t+1}^k = \alpha \hat{R}_t^k \text{ for } t \geq t_0 \tag{50}$$

$$\hat{R}_t^k = \alpha^{t-t_0}\hat{R}^{k*} \tag{51}$$

$$\tilde{R}_t^k = \alpha^{t-t_0}\left(\tilde{R}_1^{k*} - \tilde{R}_2^{k*}\right) + \tilde{R}_2^{k*} \tag{52}$$

$\alpha \in]0;1[$, the series converges monotonically to the new steady state. The sign of the convergence process is opposite to aging change. Note that if $\sigma = 1$ then $\hat{R}^* = -\hat{A}^*$

$$\tilde{R}_t^k = \tilde{R}_1^{k*} - (1-\alpha^{t-t_0})\left(\tilde{A}_2^* - \tilde{A}_1^*\right) \tag{53}$$

(ii) General case for σ and $\delta \in]0,1]$: log linearizing (49),

$$\hat{R}_{t+1}^k = (\alpha_{R^{k*},\sigma})\hat{R}_t^k \text{ for } t \geq t_0 \tag{54}$$

$$\hat{R}_t^k = (\alpha_{R^{k*},\sigma})^{t-t_0}\hat{R}^{k*} \tag{55}$$

$$\tilde{R}_t^k = (\alpha_{R^{k*},\sigma})^{t-t_0}\left(\tilde{R}_1^{k*} - \tilde{R}_2^{k*}\right) + \tilde{R}_2^{k*} \tag{56}$$

$$\text{where } \alpha_{R^{k*},\sigma} = \alpha\frac{1+\beta_{R^{k*}}}{1+\beta_{R^{k*}} + (1-\alpha)\left(\frac{1}{\sigma}-1\right)\frac{R^{k*}}{R^{k*}+(1-\delta)}} \in]0;1[\tag{57}$$

the series always converges monotonically to the new steady state. The sign of the convergence process is opposite to aging change. The convergence process takes longer for higher level of σ and lower levels of δ.

F Aggregate Demand

(i) Consumption function

From the Euler equation (10) and budget constraint of the old (16), and assuming full depreciation of capital in each period, $\delta = 1$

$$\begin{aligned} C_t &= C_t^y + C_t^o && (58) \\ &= N_t^y \frac{s_t}{\beta_{R_t,\sigma}} + R_{t-1} N_t^o s_{t-1} && (59) \\ &= \frac{1}{1+\beta_{R_t,\sigma}} \left(w_t L_t - G_t \right) + R_t^k K_t^s && (60) \\ &= \frac{1}{1+\beta_{R_t,\sigma}} \left[(1-\alpha) Y_t - G_t \right] + \alpha Y_t && (61) \\ &= \left[\frac{(1-\alpha)}{1+\beta_{R_t,\sigma}} + \alpha \right] Y_t - \frac{1}{1+\beta_{R_t,\sigma}} G_t && (62) \end{aligned}$$

(ii) Investment function

$$I_t = K_{t+1} = \alpha \frac{Y_{t+1}}{R_{t+1}^k} = \alpha \frac{Y_{t+1}}{R_{t+1}} \tag{63}$$

(iii) Aggregate Demand

$$\begin{aligned} Y_t &= C_t + I_t + G_t && (64) \\ &= \left[\frac{(1-\alpha)}{1+\beta_{R_t,\sigma}} + \alpha \right] Y_t + \alpha \frac{Y_{t+1}}{R_{t+1}} + \frac{\beta_{R_t,\sigma}}{1+\beta_{R_t,\sigma}} G_t && (65) \end{aligned}$$

(iv) Aggregate Demand per capita

$$y_t^{pc} = \left[\frac{(1-\alpha)}{1+\beta_{R_t,\sigma}} + \alpha \right] y_t^{pc} + \left(\frac{\alpha}{R_{t+1}} \right) \left[\frac{1}{A_t} \left(\frac{1+A_t}{1+A_{t-1}} \right) \right] y_{t+1}^{pc} + \frac{\beta_{R_t,\sigma}}{1+\beta_{R_t,\sigma}} G_t^{pc} \tag{66}$$

(v) Aggregate Demand per capita in steady state

$$y^{pc} = \left[\frac{1-\alpha}{1+\beta_{R,\sigma}} + \alpha + \frac{\alpha}{A} \frac{1}{R} \right] y^{pc} + \frac{\beta_{R,\sigma}}{1+\beta_{R,\sigma}} G^{pc} \tag{67}$$

Assuming that the system is determined, and taking logs, $\tilde{y}^{pc}$ is expressed in terms of R and A

$$\tilde{y}^{pc} = -\ln \left[(1-\alpha) \frac{\beta_{R,\sigma}}{1+\beta_{R,\sigma}} - \frac{\alpha}{A} \frac{1}{R} \right] + \ln \left(\frac{\beta_{R,\sigma}}{1+\beta_{R,\sigma}} G^{pc} \right) \tag{68}$$

G Impact of aging on output per capita at the ZLB

We now assume that $i = 0$, $\Pi = R = 1$,and also that $\sigma = 1$ without loss of generality. Then an increase in aging leads unambiguously to a decrease of output per capita, and:

$$\frac{d\tilde{y}^{pc}}{dA} = - \left[(1-\alpha) \frac{\beta}{1+\beta} - \frac{\alpha}{A} \right]^{-1} \frac{\alpha}{A^2} < 0 \tag{69}$$